DECAY OF KDV SOLITONS

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Abstract. In this paper we develop a linear eigenvalue decomposition for N-soliton solutions of the Korteweg-de Vries equation and use it to obtain a new mathematical explanation of two-soliton interaction in terms of particle decay. We discover that the two soliton 'particles' or pulses which appear in each solution exchange identities upon collision and emit a dual 'ghost' particle pair in order to conserve mass and momentum.

1. Introduction

It is well known that the Korteweg-de Vries (KdV) equation,

$$u_t - 6uu_x + u_{xxx} = 0,$$

is a model for many wave related phenomena and admits a special family of localized solutions called N-solitons corresponding to reflectionless potentials (cf. [M]). Here, N denotes the number of solitons, i.e. the number of pulses or potential wells, that appear in each solution. One-solitons or solitary waves were first observed by J. Scott Russell along the Union Canal at Edinburg in 1834 (cf. [M]). Then in 1895, D.J. Korteweg and G. de Vries [KV] published their (KdV) equation as a model for these waves. However, it would require another seventy years before two-soliton interaction was observed by N.J. Zabusky and M.D. Kruskal [ZK] through numerical calculation; they reported that "solitons 'pass through' one another without losing their identity". The exact interaction of two-solitons was then determined numerically by Zabusky [Z] and soon thereafter P.D. Lax [La] gave a mathematical proof.

The idea that perhaps solitons actually bounce off each other upon collision dates back to Bowtell and Stuart ([BS]). The exchange of mass that occurs between the two colliding soliton particles then allows them to exchange their identities. More recent work advocating this viewpoint can be found in [Le] and [MC]. In order to mathematically investigate this behavior, it is desirable to isolate each particle in any given N-soliton solution. This can be achieved say by decomposing the solution into a linear sum even though the KdV equation itself is nonlinear so that the superposition principle fails to hold. To this end, various such decompositions can be found in the literature (cf. [GGKM], [HM], [S], [MC]). We shall discuss some of these decompositions in relationship to ours at the end of this paper.

In this paper, we develop a linear eigenvalue decomposition of N-soliton solutions for the Korteweg-de Vries equation and use it to obtain a new mathematical explanation of two-soliton interaction in terms of particle decay. This decomposition is obtained through a diagonalization procedure that is applied to the corresponding soliton matrix and has the effect of isolating the decay of each soliton 'particle'. For two-solitons, the interaction described by Theorem 3.3 suggests a decay phenomenon that occurs frequently in elementary particle physics: the two soliton particles split upon collision, resulting in an exchange of identities and the emission of a dual 'ghost' particle pair (cf. Figure 1). Theorem 3.4 then shows that each decay process conserves mass and momentum and supports our particle decay interpretation of soliton interaction. Interesting properties of our dual ghost particles are then described in Theorem 3.6. In fact, we like to view each ghost particle as a nonlinear 'difference' between two given soliton particles. Lastly, an explicit example is given in 3.11 to illustrate our results (cf. Figures 2-4).

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2. Soliton Particles

Let N be a positive integer and assume that the initial scattering data for $u(x,0)$, obtained through the time-independent Schrodinger equation

$$\psi_{xx} - [\lambda - u(x,0)]\psi = 0, \tag{1}$$

has only a discrete energy spectrum. This means that λ takes on a discrete set of N negative energy eigenvalues $\{\lambda_1 < \lambda_2 < ... < \lambda_N < 0\}$ with corresponding eigenfunctions $\{\psi_1, \psi_2, ..., \psi_N\}$. It is standard that we normalize these eigenfunctions and compute their normalized factors c_n commonly referred to as 'phase shifts':

$$\int_{-\infty}^{\infty} \psi_n^2 dx = 1, \qquad c_n = \lim_{x\to-\infty} e^{k_n x}\psi_n. \tag{2}$$

The initial scattering data is then used to produce the N-soliton solution of the KdV equation through the determinant formula

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det(I+A). \tag{3}$$

Here, the $N \times N$ *soliton matrix* A has entries defined by

$$A = (a_{mn}); \qquad a_{mn} = \frac{c_m c_n}{k_m + k_n} e^{(k_m+k_n)x - 4(k_m^3 + k_n^3)t}, \tag{4}$$

where the spectral parameter $k_n > 0$ is defined via the relation $\lambda_n = -k_n^2$. This solution was obtained independently in the early 1970's by C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura [GGKM], M. Wadati and M. Toda [WT] both groups by means of the inverse scattering method, and by R. Hirota [H] through his direct method.

We now turn to developing our working definition of a soliton particle. It is well known that A is symmetric and positive definite (cf. [KM],[GGKM],[WT]). This allows us to diagonalize it so that

$$B^{-1}AB = D = \begin{pmatrix} \mu_1(x,t) & 0 & ... & 0 \\ 0 & & & \\ ... & & & \\ 0 & & & \mu_N(x,t) \end{pmatrix}. \tag{5}$$

Here, $\{\mu_1 > ... > \mu_N\}$ is the (ordered) set of real positive eigenvalues of A and B is the orthogonal matrix consisting of an orthonormal basis of eigenvectors of A. It follows that we can write $u(x,t)$ in terms of $\{\mu_n\}$ which we shall refer to as *decay eigenvalues*:

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det(I+A) \tag{6}$$

$$= -2\frac{\partial^2}{\partial x^2}\log\det[B^{-1}(I+A)B] \tag{7}$$

$$= -2\frac{\partial^2}{\partial x^2}\log\det(I+D) \tag{8}$$

$$= -2\frac{\partial^2}{\partial x^2}\log\prod_{n=1}^{N}[1+\mu_n(x,t)] \tag{9}$$

$$= \sum_{n=1}^{N} -2\frac{\partial^2}{\partial x^2}\log[1+\mu_n(x,t)]. \tag{10}$$

Definition 2.1. Define

$$s_n(\nu_n) \equiv -2k_n^2 \operatorname{sech}^2(k_n\nu_n), \qquad n = 1, ..., N, \tag{11}$$

to be the n-th *soliton particle* of u where $\nu_n = x - 4k_n^2 t$ is the n-th moving frame. Then we shall refer to

$$u_n(x,t) \equiv -2\frac{\partial^2}{\partial x^2}\log[1 + \mu_n(x,t)] \tag{12}$$

as the *decay function* of s_n and to the sum $u = \sum_{n=1}^N u_n$ as derived in (10) as the *decay decomposition* of u. The results of the next section will justify our use of terminology.

3. Decay of Two-Solitons

In this section we assume $N = 2$ and investigate the asymptotic behavior of the decay functions u_1 and u_2 as a means of understanding soliton interaction. We begin by writing the matrix A explicitly in terms of the two moving frames ν_1 and ν_2:

$$A = \begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1\nu_1} & \frac{c_1c_2}{k_1+k_2}e^{k_1\nu_1+k_2\nu_2} \\ \frac{c_1c_2}{k_1+k_2}e^{k_1\nu_1+k_2\nu_2} & \frac{c_2^2}{2k_2}e^{2k_2\nu_2} \end{pmatrix}. \tag{13}$$

Denoting by $p = \mathrm{Tr}(A)$ and $q = \det(A)$, it follows that the two eigenvalues of A are given by

$$\mu_1 = \frac{1}{2}\left(p + \sqrt{p^2 - 4q}\right), \tag{14}$$

$$\mu_2 = \frac{1}{2}\left(p - \sqrt{p^2 - 4q}\right). \tag{15}$$

Definition 3.1. We define

$$A_g = \begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1\nu_g} & \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_g} \\ \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_g} & \frac{c_2^2}{2k_2}e^{2k_2\nu_g} \end{pmatrix} \tag{16}$$

to be the *ghost matrix* of A where $\nu_g = x - 4k_g^2 t$ and $k_g = (k_1^2 + k_1k_2 + k_2^2)^{1/2}$. In addition, if γ_1 and γ_2 denote the eigenvalues of A_g corresponding to μ_1 and μ_2, respectively, then we shall refer to

$$g(\nu_g) \equiv -2\frac{\partial^2}{\partial \nu_g^2}\log[\gamma_1(\nu_g)] \tag{17}$$

as the *ghost particle* and

$$\bar{g}(\nu_g) \equiv -2\frac{\partial^2}{\partial \nu_g^2}\log[\gamma_2(\nu_g)] \tag{18}$$

as the *anti-ghost particle* corresponding to the pair $\{u_1, u_2\}$.

Note that ν_g represents the moving frame of both g and $\bar{g}$ and that $4k_g^2$ represents their velocity and exceeds that of the two soliton particles. The following lemma assures us that the correspondence mentioned above between the two sets of eigenvalues is well defined.

Lemma 3.2. *Denote by* $\hat{k}^2 = k_1^2k_2 + k_1k_2^2$. *Then*

$$A = e^{8\hat{k}^2 t}A_g.$$

Moreover, $\mu_n = e^{8\hat{k}^2 t}\gamma_n$ *for* $n = 1, 2$.

Proof. It suffices to prove that every coefficient of A has $e^{8\hat{k}^2 t}$ as a common factor when rewritten in terms of ν_g. This quickly follows from the relation

$$\begin{aligned} e^{k_n(x-4k_n^2 t)} &= e^{k_n(\nu_g+4k_g^2 t-4k_n^2 t)} \\ &= e^{k_n\nu_g+4(k_1^2 k_2+k_1k_2^2)t} \\ &= e^{4\hat{k}^2 t}e^{k_n\nu_g}. \end{aligned}$$

The fact that $\mu_n = e^{8\hat{k}^2 t}\gamma_n$ also follows from this relation and can be easily checked by the reader. $\square$

We are now ready to present our theorem describing particle decay of two-solitons. This will justify our use of the terms 'particle' and 'decay' in referring to s_n and u_n, respectively.

Theorem 3.3. *The following asymptotic relations hold for u_1 and u_2:*
(i)

$$\begin{aligned} u_1 &\sim s_1(\nu_1+\delta_1), && \textit{as } t\to-\infty \\ u_1 &\sim s_2(\nu_2+\delta_2)+g(\nu_g), && \textit{as } t\to\infty \end{aligned}$$

in the sense that

$$\lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} u_1 = s_1(\nu_1+\delta_1), \quad \lim_{\substack{\nu_2 \text{ fixed} \\ t\to\infty}} u_1 = s_2(\nu_2+\delta_2), \quad \lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} u_1 = g(\nu_g).$$

Here, the relative phase shifts δ_1 and δ_2 are defined by

$$e^{2k_1\delta_1} = \frac{c_1^2}{2k_1}, \qquad e^{2k_2\delta_2} = \frac{c_2^2}{2k_2}.$$

(ii)

$$\begin{aligned} u_2 &\sim s_2(\nu_2+\delta_2+\Delta), && \textit{as } t\to-\infty \\ u_2 &\sim s_1(\nu_1+\delta_1+\Delta)+\bar{g}(\nu_g), && \textit{as } t\to\infty \end{aligned}$$

Here, Δ is defined by

$$e^{2k_2\Delta} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}.$$

Following the physics literature we shall summarize the decay described by u_1 and u_2 as follows:

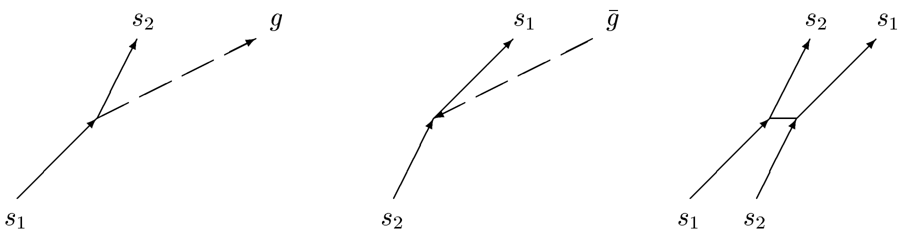
$$\begin{aligned} u_1: &\quad s_1 \to s_2+g, \\ u_2: &\quad s_2 \to s_1+\bar{g}. \end{aligned}$$

The corresponding space-time plots are drawn in Figure 1. Notice that they describe the exchange of identities between s_1 and s_2 and the fact that the emitted ghost particles (represented by the dashed lines) have velocities greater than both soliton particles.

Proof of Theorem 3.3. (i) Our approach is to analyze u_1 from the perspective of the three moving frames corresponding to the velocities ν_1, ν_2 and ν_g and to treat each as a separate case:

CASE I: Assume ν_1 is fixed. We rewrite the trace and determinant of A as

$$\begin{aligned} p &= \mathrm{Tr}(A) \\ &= \tfrac{c_1^2}{2k_1}e^{2k_1\nu_1} + \tfrac{c_2^2}{2k_2}e^{2k_2\nu_2} \\ &= e^{2k_1\nu_1}\left(\tfrac{c_1^2}{2k_1} + \tfrac{c_2^2}{2k_2}e^{2k_2\nu_2-2k_1\nu_1}\right) \\ &= e^{2k_1\nu_1}\left(\tfrac{c_1^2}{2k_1} + \tfrac{c_2^2}{2k_2}e^{2(k_2-k_1)\nu_1+8k_2(k_1^2-k_2^2)t}\right) \end{aligned}$$



(a) Diagram of u_1 (b) Diagram of u_2 (c) Diagram of u

FIGURE 1. Space-time plots of two-soliton decay

and

$$\begin{aligned} q &= \det(A) \\ &= \left(\tfrac{k_1-k_2}{k_1+k_2}\right)^2 \tfrac{c_1^2c_2^2}{4k_1k_2} e^{2(k_1\nu_1+k_2\nu_2)} \\ &= e^{2k_1\nu_1}\left(\tfrac{k_1-k_2}{k_1+k_2}\right)^2 \tfrac{c_1^2c_2^2}{4k_1k_2} e^{2k_2\nu_1+8(k_1^2-k_2^2)t} \end{aligned}$$

so that

$$\lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} p = \frac{c_1^2}{2k_1}e^{2k_1\nu_1}, \qquad \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} q = 0.$$

This forces

$$\begin{aligned} \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} (1+\mu_1) &= \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} \left[1+\frac{1}{2}\left(p+\sqrt{p^2-4q}\right)\right] \\ &= 1+\tfrac{c_1^2}{2k_1}e^{2k_1\nu_1} \end{aligned}$$

and implies

$$\begin{aligned} \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} u_1 &= \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(1+\mu_1)\right\} \\ &= -2\tfrac{\partial^2}{\partial x^2}\log(1+\tfrac{c_1^2}{2k_1}e^{2k_1\nu_1}) \\ &= \frac{-8k_1c_1^2e^{2k_1\nu_1}}{\left(1+\frac{c_1^2}{2k_1}e^{2k_1\nu_1}\right)^2} \\ &= s_1(\nu_1+\delta_1) \end{aligned}$$

where δ_1 is defined by $e^{2k_1\delta_1} = \frac{c_1^2}{2k_1}$. Note that we have implicitly used the fact $\frac{\partial}{\partial x} = \frac{\partial}{\partial \nu_1}$.

CASE II: Assume that ν_2 is fixed. We proceed in the same manner as CASE I but factor $e^{2k_2\nu_2}$ instead of $e^{2k_1\nu_1}$ from p and q. It is then a straightforward exercise to show that

$$\lim_{\substack{\nu_2 \text{ fixed} \\ t\to\infty}} u_1 = s_2(\nu_2+\delta_2)$$

where this time δ_2 is defined by $e^{2k_2\delta_2} = \frac{c_2^2}{2k_2}$.

CASE III: Assume ν_g is fixed. Applying Lemma 3.2, we obtain for $t \to -\infty$:

$$\begin{aligned}\lim_{\substack{\nu_g \text{ fixed} \\ t\to-\infty}} u_1 &= \lim_{\substack{\nu_g \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(1+\mu_1)\right\} \\ &= \lim_{\substack{\nu_g \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(1+e^{4\hat{k}^2 t}\gamma_1)\right\} \\ &= 0.\end{aligned}$$

On the other hand for $t \to \infty$:

$$\begin{aligned}\lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} u_1 &= \lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(1+e^{4\hat{k}^2 t}\gamma_1)\right\} \\ &= \lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log e^{4\hat{k}^2 t} + \log(e^{-4\hat{k}^2 t}+\gamma_1)\right\} \\ &= \lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(e^{-4\hat{k}^2 t}+\gamma_1)\right\} \\ &= -2\tfrac{\partial^2}{\partial \nu_g^2}\log\gamma_1 \\ &= g.\end{aligned}$$

This completes the proof of part (i).

(ii) We apply a similar analysis to u_2 by again considering three separate cases:

CASE I: Assume ν_2 is fixed. We rewrite p and q as

$$\begin{aligned}p &= e^{8k_1(k_2^2-k_1^2)t}\left(\tfrac{c_1^2}{2k_1}e^{2k_1\nu_2} + \tfrac{c_2^2}{2k_2}e^{2k_2\nu_2-8k_1(k_2^2-k_1^2)t}\right) \\ q &= e^{8k_1(k_2^2-k_1^2)t}\tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}\tfrac{c_1^2c_2^2}{4k_1k_2}e^{2(k_1+k_2)\nu_2}.\end{aligned}$$

The relations

$$\lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \frac{q}{p} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2}, \qquad \lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \frac{q}{p^2} = 0$$

now tell us how μ_2 behaves in the limit once we rationalize it:

$$\begin{aligned}\lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \mu_2 &= \lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \left\{\frac{1}{2}\left(p-\sqrt{p^2-4q}\right)\frac{p+\sqrt{p^2-4q}}{p-\sqrt{p^2-4q}}\right\} \\ &= \lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \left\{\frac{\frac{2q}{p}}{1+\sqrt{1-\frac{4q}{p^2}}}\right\} \\ &= \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}\tfrac{c_2^2}{2k_2}e^{2k_2\nu_2}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} u_2 &= \lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log(1+\mu_2)\right\} \\ &= \lim_{\substack{\nu_2 \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2}\right]\right\} \\ &= s_2(\nu_2+\delta_2+\Delta)\end{aligned}$$

where Δ is defined by $e^{2k_2\Delta} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$.

CASE II: Assume that ν_1 is fixed. As the line of argument here is the same as that for CASE I with ν_2 fixed, we leave it for the reader to verify that

$$\lim_{\substack{\nu_1 \text{ fixed} \\ t\to\infty}} u_2 = s_1(\nu_1+\delta_1+\Delta).$$

CASE III: Assume that ν_g is fixed. The proof of

$$\lim_{\substack{\nu_g \text{ fixed} \\ t\to\infty}} u_2 = \bar{g}$$

is exactly the same as that for CASE III in (i) and will be left for the reader. This completes the proof of our theorem. □

The following result provides evidence to support our theory of soliton decay.

Theorem 3.4. *(i) Conservation of mass:*

$$\int_{-\infty}^{\infty} u_n(x,t)dx = -4k_n, \qquad n=1,2.$$

(ii) Conservation of momentum:

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu_n(x,t)dx = -16k_n^3, \qquad n=1,2.$$

Proof. (i) For u_1, we have

$$\begin{aligned}
\textstyle\int_{-\infty}^{\infty} u_1(x,t)dx &= \textstyle\int_{-\infty}^{\infty}\left[-2\frac{\partial^2}{\partial x^2}\log(1+\mu_1)\right]dx \\
&= \left[-2\tfrac{\partial}{\partial x}\log(1+\mu_1)\right]_{-\infty}^{\infty} \\
&= -2\left[\tfrac{\mu_1'}{1+\mu_1}\right]_{-\infty}^{\infty} \\
&= -4k_1.
\end{aligned}$$

A similar argument applied to u_2 (after first rationalizing μ_2) shows that $\int_{-\infty}^{\infty} u_2(x,t)dx = -4k_2$.

(ii) Integration by parts yields

$$\begin{aligned}
\textstyle\int_{-\infty}^{L} xu_n(x,t)dx &= \left[-2x\tfrac{\partial}{\partial x}\log(1+\mu_n)\right]_{-\infty}^{L} - \textstyle\int_{-\infty}^{L}\left[-2\frac{\partial}{\partial x}\log(1+\mu_n)\right]dx \\
&= -2L\tfrac{\mu_n'(L)}{(1+\mu_n(L)} + 2\log(1+\mu_n(L)) \\
&\sim -4k_nL + 4k_n(L-4k_n^2t+\delta_n)
\end{aligned}$$

as $L\to\infty$. It follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu_n(x,t)dx = -16k_n^3, \qquad n=1,2.$$

□

For $n=1,2$, we define the *center of mass* of u_n to be

$$x_n(t) \equiv \frac{\int_{-\infty}^{\infty} xu_n(x,t)dx}{\int_{-\infty}^{\infty} u_n(x,t)dx}. \tag{19}$$

It follows immediately from Theorem 3.4 that

Corollary 3.5. *The center of mass $x_n(t)$ as defined by (19) moves with constant velocity $4k_n^2$, i.e.*

$$\frac{dx_n}{dt} = 4k_n^2, \qquad n=1,2.$$

Let us now investigate our ghost particles a little more closely. We begin with the following theorem which justifies our use of the terms 'ghost' and 'anti-ghost' for g and $\bar{g}$ as they do not appear in u due to cancellation.

Theorem 3.6. *The ghost particles g and $\bar{g}$ enjoy the following properties:*
(i) $g + \bar{g} = 0$.
(ii) $\int_{-\infty}^{\infty} g(\nu_g)d\nu_g = 4(k_1 - k_2)$.
(iii) $g = -32k_1k_2\left(\frac{p_g q_g}{r_g^{3/2}}\right) < 0$, *where* $p_g = \mathrm{Tr}(A_g)$, $q_g = \det(A)$ *and* $r_g = p_g^2 - 4q_g$.
(iv) $g(\nu_g) = O(\mathrm{sech}^2[(k_1 - k_2)(\nu_g + \delta_g)])$ *as* $\nu_g \to \pm\infty$, *where* δ_g *is defined by* $e^{2(k_1-k_2)\delta_g} = \frac{c_1^2 k_2}{c_2^2 k_1}$.
(v) $|g(\nu_g)| \leq \frac{(k_1 - k_2)^2(k_1 + k_2)}{\sqrt{k_1 k_2}}$ *with equality holding precisely when* $\nu_g = -\delta_g$.

Proof. (i) If one recalls that

$$\begin{aligned} \gamma_1\gamma_2 &= \det(A_g) \\ &= \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}\tfrac{c_1^2c_2^2}{4k_1k_2}e^{2(k_1+k_2)\nu_g}, \end{aligned}$$

then it directly follows

$$\begin{aligned} g + \bar{g} &= -2\tfrac{\partial^2}{\partial \nu_g^2}\log(\gamma_1\gamma_2) \\ &= 0. \end{aligned}$$

(ii) We have

$$\begin{aligned} \textstyle\int_{-\infty}^{\infty} g(\nu_g)d\nu_g &= \textstyle\int_{-\infty}^{\infty}\left[-2\tfrac{\partial^2}{\partial\nu_g^2}\log\gamma_1\right]d\nu_g \\ &= \left[-2\tfrac{\gamma_1'}{\gamma_1}\right]_{-\infty}^{\infty}. \end{aligned} \tag{20}$$

Substituting the relations

$$\lim_{\nu_g\to-\infty}\frac{\gamma_1'}{\gamma_1} = 2k_2, \qquad \lim_{\nu_g\to\infty}\frac{\gamma_1'}{\gamma_1} = 2k_1.$$

into (20) then yields the desired result:

$$\int_{-\infty}^{\infty} g(\nu_g)d\nu_g = 4(k_2 - k_1).$$

We note that this result also follows directly from Theorem 3.4 due to conservation of mass of u_1.

(iii) First write γ_1 in the form

$$\gamma_1 = \frac{1}{2}\left(p_g + \sqrt{r_g}\right) \tag{21}$$

where

$$p_g = \mathrm{Tr}(A_g) = \frac{c_1^2}{2k_1}e^{2k_1\nu_g} + \frac{c_2^2}{2k_2}e^{2k_2\nu_g}, \tag{22}$$

$$q_g = \det(A_g) = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\frac{c_1^2c_2^2}{4k_1k_2}e^{2(k_1+k_2)\nu_g}, \tag{23}$$

$$r_g = p_g^2 - 4q_g. \tag{24}$$

Then we can express γ_1 in terms of an appropriate hyperbolic cosine function by introducing the identity

$$p_g = \frac{c_1 c_2}{\sqrt{k_1 k_2}} e^{(k_1+k_2)\nu_g} \cosh[(k_1 - k_2)(\nu_g + \delta_g)], \tag{25}$$

where δ_g is defined by the relation $e^{2(k_1-k_2)\delta_g} = \frac{c_1^2 k_2}{c_2^2 k_1}$. It follows that

$$\gamma_1 = \frac{c_1 c_2}{\sqrt{k_1 k_2}} e^{(k_1+k_2)\nu_g} \left(\cosh\left[(k_1 - k_2)(\nu_g + \delta_g)\right] + \sqrt{\cosh^2\left[(k_1 - k_2)(\nu_g + \delta_g)\right] - \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}} \right) \tag{26}$$

$$= \frac{(k_1 - k_2)}{(k_1 + k_2)} \frac{c_1 c_2}{\sqrt{k_1 k_2}} e^{(k_1+k_2)\nu_g} \left(z + \sqrt{z^2 - 1} \right), \tag{27}$$

where $z = \frac{(k_1+k_2)}{(k_1-k_2)} \cosh\left[(k_1 - k_2)(\nu_g + \delta_g)\right]$. Therefore,

$$g = -2 \frac{\partial^2}{\partial \nu_g^2} \log \gamma_1 \tag{28}$$

$$= -2 \frac{\partial^2}{\partial \nu_g^2} \left[\log\left(\frac{(k_1 k_2)}{(k_1 + k_2)} \frac{c_1 c_2}{\sqrt{k_1 k_2}} e^{(k_1+k_2)\nu_g} \right) + \log(z + \sqrt{z^2 - 1}) \right] \tag{29}$$

$$= -2 \frac{\partial^2}{\partial \nu_g^2} \cosh^{-1} z \tag{30}$$

$$= -8 k_1 k_2 \frac{z}{(z^2 - 1)^{3/2}} \tag{31}$$

$$= -8 k_1 k_2 \frac{\frac{(k_1+k_2)}{(k_1-k_2)} \cosh\left[(k_1 - k_2)(\nu_g + \delta_g)\right]}{\left[\frac{(k_1+k_2)^2}{(k_1-k_2)^2} \cosh^2\left[(k_1 - k_2)(\nu_g + \delta_g)\right] - 1\right]^{3/2}} \tag{32}$$

$$= -32 k_1 k_2 \frac{p_g q_g}{r_g^{3/2}}, \tag{33}$$

as desired. Moreover, g is negative because the quantities p_g, q_g, and r_g are all positive.

(iv) It is now easy to deduce from (32) that

$$g(\nu_g) = \mathrm{O}(\mathrm{sech}^2[(k_1 - k_2)(\nu_g + \delta_g)])$$

as $\nu_g \to \pm\infty$.

(v) Using (31), we find that $g(\nu_g)$ has derivative

$$\frac{dg}{d\nu_g} = 8 k_1 k_2 \frac{2z^2 + 1}{(z^2 - 1)^{3/2}} \left(\frac{dz}{d\nu_g} \right). \tag{34}$$

Since $z^2 - 1 > 0$, it follows that $\frac{dg}{d\nu_g}$ is zero precisely when

$$\frac{dz}{d\nu_g} = \frac{(k_1 + k_2)^2}{(k_1 - k_2)} \sinh\left[(k_1 - k_2)(\nu_g + \delta_g)\right] \tag{35}$$

is zero, or equivalently, when $\nu_g = -\delta_g$. We can therefore conclude that g has an absolute minimum of

$$g(-\delta_g) = -\frac{(k_1 - k_2)^2 (k_1 + k_2)}{\sqrt{k_1 k_2}}$$

at this critical point because of (iv). This completes the proof of Theorem 3.6. □

Remark 3.7. We remark that property (iv) of Theorem 3.6 shows that in some sense g can be viewed as a nonlinear difference between the soliton particles s_1 and s_2 as defined by (11). Moreover, $g(\nu_g) \to 0$ as $k_2 \to k_1$ and $g(\nu_g) \to -4k_1\delta(\nu_g)$ as $k_2 \to 0$, where $\delta(\nu_g)$ is the Dirac delta function.

Next, we show that each decay function itself can be decomposed as a sum of a 'soliton' term and a 'ghost' term:

$$u_n(x,t) = -2\frac{\partial^2}{\partial x^2}\log(1+\mu_n) \tag{36}$$

$$= -2\left[\frac{(1+\mu_n)\mu_n'' - (\mu_n')^2}{(1+\mu_n)^2}\right] \tag{37}$$

$$= -2\frac{\mu_n''}{(1+\mu_n)^2} - 2\left[\frac{\mu_n\mu_n'' - (\mu_n')^2}{\mu_n^2}\right]\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{38}$$

$$= -2\frac{\mu_n''}{(1+\mu_n)^2} - 2\left(\frac{\partial^2}{\partial x^2}\log\mu_n\right)\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{39}$$

$$= u_n^s + u_n^g. \tag{40}$$

Definition 3.8. We shall call

$$u_n^s = -2\frac{\mu_n''}{(1+\mu_n)^2} \tag{41}$$

the *soliton component* of u_n and

$$u_n^g = -2\left(\frac{\partial^2}{\partial x^2}\log\mu_n\right)\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{42}$$

the *ghost component* of u_n. Morevover, we shall refer to the decomposition given by (40) as the *splitting decomposition* of u_n.

For two-solitons, it follows that

Corollary 3.9.

$$u_n^g(x,t) = (-1)^{n-1}g(x-4k_g^2t)\left(\frac{\mu_n(x,t)}{1+\mu_n(x,t)}\right)^2, \qquad n = 1,2. \tag{43}$$

Remark 3.10. The decomposition described in (40) reveals mathematically the time-asymmetry of soliton decay in that ghost particles are born at $t = \infty$ and is essentially due to the identity matrix appearing in the N-soliton formula. In particular, the behavior of $\mu_n/(1+\mu_n) \to 0$ as $t \to -\infty$ and $\mu_n/(1+\mu_n) \to 1$ as $t \to \infty$ in (43) indicates that the ghost component u_n^g represents creation of the ghost particle $g(x - 4k_g^2 t)$ at $t = \infty$. This implies that there is actually interaction between solitons even before 'collision' occurs; however, this interaction is insignificant until then. Lastly, it is straightforward to verify that each soliton component u_n^s asymptotically describes an exchange of identities between the two soliton particles.

We end our paper with a concrete example to illustrate our results.

Example 3.11. Let $k_1 = c_1 = 2$ and $k_2 = c_2 = 1$ be the given scattering data. Our soliton matrix A then takes the form

$$A = \begin{pmatrix} e^{4x-64t} & \frac{2}{3}e^{3x-36t} \\ \frac{2}{3}e^{3x-36t} & \frac{1}{2}e^{2x-8t} \end{pmatrix} \tag{44}$$

and has eigenvalues

$$\mu_1 = \frac{1}{12}\left(3e^{2x-8t} + 6e^{4x-64t} + e^{2x-8t}\sqrt{9 + 28e^{2x-56t} + 36e^{4x-112t}}\right), \tag{45}$$

$$\mu_2 = \frac{1}{12}\left(3e^{2x-8t} + 6e^{4x-64t} - e^{2x-8t}\sqrt{9 + 28e^{2x-56t} + 36e^{4x-112t}}\right). \tag{46}$$

The decay functions u_1 and u_2 can now of course be computed through the formula

$$u_n = -2\frac{\partial^2}{\partial x^2}\log(1+\mu_n), \qquad n = 1, 2$$

but we shall avoid doing this here due to their complicated expressions.

The ghost matrix

$$A_g = \begin{pmatrix} e^{4\nu_g} & \frac{2}{3}e^{3\nu_g} \\ \frac{2}{3}e^{3\nu_g} & \frac{1}{2}e^{2\nu_g} \end{pmatrix} \tag{47}$$

has eigenvalues

$$\gamma_1 = \frac{1}{12}\left(3e^{2\nu_g} + 6e^{4\nu_g} + e^{2\nu_g}\sqrt{9 + 28e^{2\nu_g} + 36e^{4\nu_g}}\right), \tag{48}$$

$$\gamma_2 = \frac{1}{12}\left(3e^{2\nu_g} + 6e^{4\nu_g} - e^{2\nu_g}\sqrt{9 + 28e^{2\nu_g} + 36e^{4\nu_g}}\right). \tag{49}$$

Therefore,

$$g = -32k_1k_2\left(\frac{p_g q_g}{r_g^{3/2}}\right) \tag{50}$$

$$= -\frac{384e^{2\nu_g}(1+2e^{2\nu_g})}{(9 + 28e^{2\nu_g} + 36e^{4\nu_g})^{3/2}} \tag{51}$$

$$= -\frac{48\cosh(\nu_g + \log\sqrt{2})}{\left[9\cosh^2(\nu_g + \log\sqrt{2}) - 1\right]^{3/2}} \tag{52}$$

and the ghost moving frame is given by $\nu_g = x - 28t$. Of course, we also have $\bar{g} = -g$.

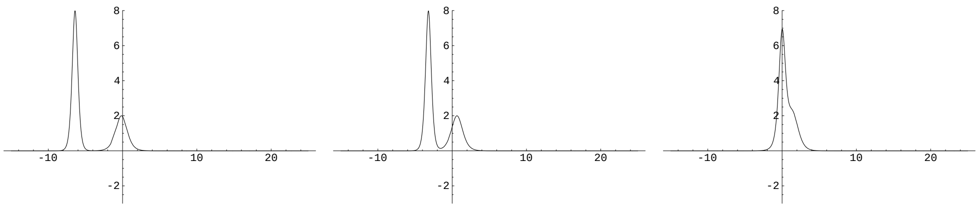
Figures 2-4 illustrate the motions of $-u(x,t)$, $-u_1(x,t)$ and $-u_2(x,t)$, respectively, over time through a sequence of six frames corresponding to $t = -0.4, -0.2, ..., 0.6$. The soliton particles s_1 and s_2 have amplitudes of 8 and 2, respectively, and velocities of 16 and 4, respectively. The ghost particle g has an amplitude of $3/\sqrt{2} \approx 2.12$ and a velocity of 28. Splitting occurs in the fourth frame at $t = 0.2$ for both u_1 and u_2 as seen in Figures 3 and 4, respectively.

Concluding Remarks. Our work raises interesting questions some of which deserve comment:

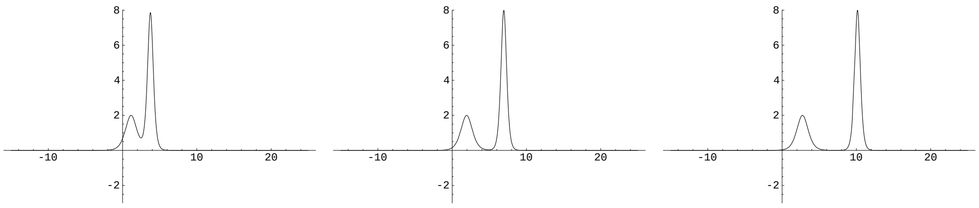
Q1. What happens during collisions of more than two solitons? Are more ghost particles generated? Can ghost particles from different pairs interact?

A1. It is found that each collision between any two soliton particles produces a ghost particle pair with the same properties as those described by Theorem 3.6. On the other hand, each collision between two ghost particles where each comes from a different pair will result in their fusion. Because of duality, there is an accompanying fission process which is interpreted as the same fusion process but reversed in time. Moreover, the final states of all ghost particles created is independent of their order of collision (modulo phase shifts). A mathematical theory formulating the creation and interaction of ghost particles will be described in a forthcoming paper.

Q2. Do the decay functions $\{u_n\}$ satisfy any partial differential equations?

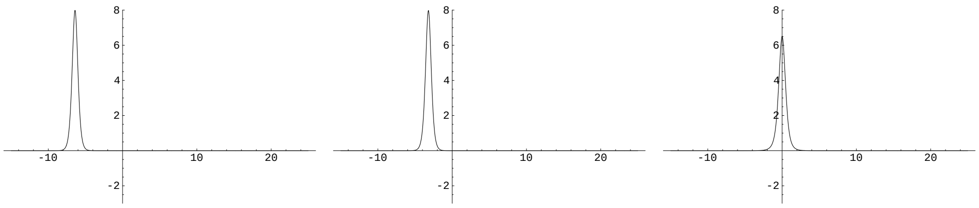


(a) $t = -0.4$ (b) $t = -0.2$ (c) $t = 0$

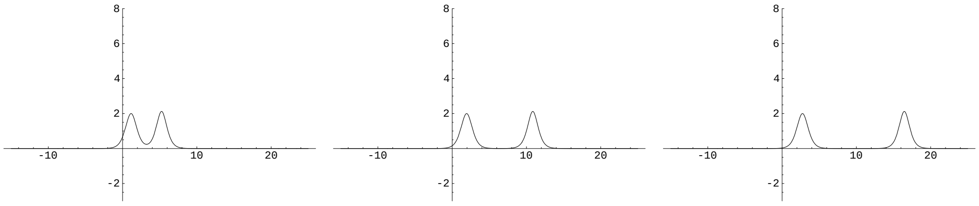


(d) $t = 0.2$ (e) $t = 0.4$ (f) $t = 0.6$

FIGURE 2. Plots of $-u(x,t)$

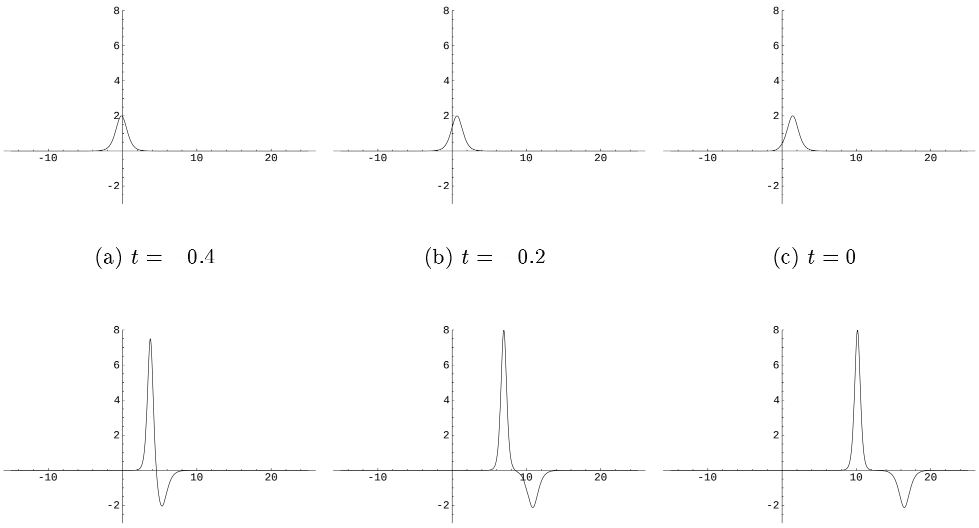


(a) $t = -0.4$ (b) $t = -0.2$ (c) $t = 0$



(d) $t = 0.2$ (e) $t = 0.4$ (f) $t = 0.6$

FIGURE 3. Plots of $-u_1(x,t)$: $s_1 \to s_2 + g$.



(a) $t = -0.4$ (b) $t = -0.2$ (c) $t = 0$

(d) $t = 0.2$ (e) $t = 0.4$ (f) $t = 0.6$

FIGURE 4. Plots of $-u_2(x,t)$: $s_2 \to s_1 + \bar{g}$.

A2. This is not presently known as we have been unsuccessful at finding such equations. On the other hand, it is known that the eigenvalues $\{\mu_n\}$ of the soliton matrix A which defines $\{u_n\}$ satisfy ordinary differential equations of the form

$$\frac{d\mu_n}{dx} = (E^T \cdot X_n)^2, \qquad n = 1, ..., N. \tag{53}$$

Here, X_n is the eigenvector of A corresponding to μ_n and $E^T = (c_1 e^{k_1\nu_1}, c_2 e^{k_2\nu_2}, ..., c_N e^{k_1\nu_1})^T$. These differential equations can be easily derived from the symmetry and positive definiteness of A. However, their usefulness is unclear as they do not make direct use of the KdV equation.

Q3. How is the linear eigenvalue decomposition described in this paper related to others in the literature, e.g. Hodnett-Moloney [HM] and Miller-Christiansen [MC]?

A3. Hodnett-Moloney's work in [HM] involves using the Hirota formalism to decompose each N-soliton solution into a linear sum of squares of hyperbolic secant functions having time-dependent amplitudes and phase shifts (a Lie-theoretic generalization of this decomposition is given by Fuchssteiner in [F]). For two-solitons, this decomposition takes the form

$$u = u_1 + u_2, \tag{54}$$

where

$$u_1 = 2a_1^2 H(\theta_2)\operatorname{sech}^2[\theta_1 + G(\theta_2)], \tag{55}$$

$$u_2 = 2a_2^2 H(\theta_1)\operatorname{sech}^2[\theta_2 + G(\theta_1)]. \tag{56}$$

Here, a_i and θ_i are the spectral parameters and moving frames, respectively. Exact formulas for $H(\nu_1)$ and $G(\nu_2)$ can then be derived by requiring u_1 and u_2 to conserve mass for all times as in Theorem 3.4. In essence, this approach views the secant function as the building block for a soliton particle whereas our approach views the eigenvalues of the soliton matrix A as the building block. As a result, the decomposition of Hodnett-Moloney seems to asymptotically describe only an exchange of soliton identities and not soliton decay as revealed by our decomposition.

As for Miller-Christiansen [MC], they considered soliton solutions of the coupled system

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x}\left[\frac{u_k}{2}\sum_{j=1}^{N} u_j + \frac{\partial^2 u_k}{\partial x^2}\right] = 0, \qquad k = 1, ..., N. \tag{57}$$

This system can be viewed as a multicomponent generalization of the KdV equation and is derived by requiring symmetry and conservation of mass principles. For $N = 2$, numerical solutions for u_1 and u_2 were obtained which indicated an exchange of mass between two given soliton particles after collision. However, there is no prediction of ghost particles which again is in contrast to our decomposition. In short, we believe our model of soliton interaction to be one that is most consistent with the laws of classical mechanics.

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