TRAVELING WAVE SOLUTIONS TO THE MOLECULAR LASER EQUATIONS

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Abstract. We discuss traveling wave solutions to the molecular laser equations.

1. Introduction

The molecular laser is the doubly-massive analog of the atomic laser that is based on the principles of Bose-Enstein condensates (BEC) and Feshbach resonance. In [LS], Hong Ling describes a mathematical model which we call the molecular laser equations (MLE) (see also [DKH]):

$$i\frac{\partial\phi_a}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi_a}{\partial x^2} + (\lambda_a|\phi_a|^2 + \lambda|\phi_m|^2)\phi_a + V\phi_a + \alpha\phi_m\phi_a^* \tag{1}$$

$$i\frac{\partial\phi_m}{\partial t} = -\frac{1}{4}\frac{\partial^2\phi_m}{\partial x^2} + (\lambda|\phi_a|^2 + \lambda_m|\phi_m|^2)\phi_m + (V+\epsilon)\phi_m + \frac{\alpha}{2}\phi_a^2 \tag{2}$$

Here, ϕ_a and ϕ_m are the atomic and molecular fields, respectively, V and ϵ are the strength of external magnetic fields, λ_a, λ_m, and and λ are the strengths of atomic, molecular, and atomic-molecular interactions, respectively, and α is the atom-to-molecule rate of conversion.

In this paper we discuss certain traveling wave solutions and small amplitude approximations to MLE. To this end, we shall assume that ϕ_a and ϕ_m have the form

$$\phi_a(x,t) = u(x-ct)e^{i(kt+\omega x)} \tag{3}$$

$$\phi_m(x,t) = v(x-ct)e^{2i(kt+\omega x)} \tag{4}$$

where u and v are real and c, k, l, θ are constants. Denoting the moving frame here by $z = x - ct$, we obtain the following formulas:

$$i\frac{\partial\phi_a}{\partial t} = -icu_ze^{i\theta} - kue^{i\theta} \tag{5}$$

$$-\frac{1}{2}\frac{\partial^2\phi_a}{\partial x^2} = -\frac{1}{2}\frac{\partial}{\partial x^2}\left(u_ze^{i\theta} + i\omega ue^{i\theta}\right) \tag{6}$$

$$= -\frac{1}{2}\left(u_{zz}e^{i\theta} + 2i\omega u_ze^{i\theta} - \omega^2ue^{i\theta}\right) \tag{7}$$

Substituting these formulas into (1) and (2) and separating real and imaginary parts leads to

$$u_t = -ku_x \Rightarrow u(x,t) = u(x-kt) \Rightarrow c = k \tag{8}$$

$$v_t = -kv_x \Rightarrow v(x,t) = v(x-kt) \tag{9}$$

$$\frac{1}{2}u_{xx} = (\frac{1}{2}k^2 + \omega + V)u + \lambda_au^3 + \lambda v^2u + \alpha vu \tag{10}$$

$$\frac{1}{4}v_{xx} = (k^2 + 2\omega + \epsilon + V)v + \lambda u^2v + \lambda_mv^3 + \frac{\alpha}{2}u^2 \tag{11}$$

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2. Linear Solutions

In this section we assume that u and v are proportional, i.e. $v = Au$, where A is constant. Then the second half of the equations above become

$$\frac{1}{2}u_{xx} = (\frac{1}{2}k^2 + \omega + V)u + \lambda_a u^3 + \lambda A^2 u^3 + \alpha A u^2 \tag{12}$$

$$\frac{1}{4}Au_{xx} = (k^2 + 2\omega + \epsilon + V)Au + \lambda A u^3 + \lambda_m A^3 u^3 + \frac{\alpha}{2}u^2 \tag{13}$$

Equating coefficients yields

$$A\alpha = \alpha/A \Rightarrow A = \pm 1 \tag{14}$$

$$\lambda_a + \lambda A^2 = 2\lambda + 2\lambda_m A^2 \Rightarrow \lambda_a = \lambda + 2\lambda_m \tag{15}$$

$$\frac{1}{2}k^2 + \omega + V = 2(k^2 + 2\omega + \epsilon + V) \Rightarrow \omega = -\left(\frac{k^2}{2} + \frac{2\epsilon + V}{3}\right) \tag{16}$$

Assuming these restrictions, we can then integrate either equation to obtain

$$\frac{1}{2}u_{xx} = \frac{2}{3}(V - \epsilon)u + \alpha A u^2 + (\lambda_a + \lambda)u^3 \tag{17}$$

$$\Rightarrow u_x^2 = C + au^2 + bu^3 + (\lambda_a + \lambda)u^4 \tag{18}$$

where $a = \frac{4}{3}(V - \epsilon)$ and $b = \frac{4}{3}\alpha A$. Lastly, we separate variables and integrate to solve for u.

$$\int \frac{du}{\sqrt{C + au^2 + bu^3 + (\lambda_a + \lambda)u^4}} = \int dx \tag{19}$$

We now concentrate on cases which yield bounded solutions:

CASE I: $C = 0$, $a > 0$, $b < 0$. It follows that

$$u(x) = \frac{4a^{3/2}e^{\sqrt{a}x}}{a - 2\sqrt{a}be^{\sqrt{a}x} + b^2 e^{2\sqrt{a}x} - 4a(\lambda_a + \lambda)e^{2\sqrt{a}x}} \tag{20}$$

CASE II: $C = 0$, $\lambda_a + \lambda = 0$, $a > 0$, $b < 0$. In this case,

$$u(x) = -\frac{a}{b}\operatorname{sech}^2\left[\frac{\sqrt{a}}{2}\,(x + \delta)\right] \tag{21}$$

and so

$$\phi_a(x,t) = \pm\frac{\epsilon - V}{\alpha}\operatorname{sech}^2\left[\sqrt{\frac{V - \epsilon}{3}}(x - kt + \delta)\right]\exp\left\{i\left[kt - \left(\frac{k^2}{2} + \frac{V + 2\epsilon}{3}\right)x\right]\right\} \tag{22}$$

$$\phi_m(x,t) = \mp\frac{\epsilon - V}{\alpha}\operatorname{sech}^2\left[\sqrt{\frac{V - \epsilon}{3}}(x - kt + \delta)\right]\exp\left\{2i\left[kt - \left(\frac{k^2}{2} + \frac{V + 2\epsilon}{3}\right)x\right]\right\} \tag{23}$$

$$(-k + c^2)u = -\frac{1}{2}u_{zz} + \frac{1}{2}c^2 u + (\lambda_a u^2 + \lambda v^2)u + \omega u + \alpha uv \tag{24}$$

$$(-2k + 2c^2)v = -\frac{1}{4}v_{zz} + c^2 v + (\lambda u^2 + \lambda_m v^2)v + (\omega + \epsilon)v + \frac{\alpha}{2}u^2 \tag{25}$$

3. Quadratic Solutions

In this section we further assume that $u > 0$ and $v = Ku^2$ with $K > 0$. Equations (24)-(25) above then simplify to

$$u_{zz} = au + bu^3 + pu^5 \tag{26}$$

$$K(u_z)^2 = -Ku_{zz}u + du^2 + eu^4 + qu^6 \tag{27}$$

where

$$a = 2(k+\omega) - c^2, \quad b = 2(\lambda_a + \alpha K), \quad p = 2\lambda K^2 \tag{28}$$

$$d = 2(2k + \omega + \epsilon - c^2)K + \alpha, \quad e = 2\lambda, \quad q = 2\lambda_m K^3 \tag{29}$$

Integrating (26) yields

$$u_z^2 = au^2 + \frac{b}{2}u^4 + \frac{p}{3}u^6 + C_1 \tag{30}$$

On the other hand, substituting (26) into (27) produces

$$u_z^2 = -C_1 u + (d/K - a)u^2 + (e/K - b)u^4 + (q/K - p)u^6 \tag{31}$$

In order for (30) and (31) to be consistent with each other, we therefore require that the following constraints hold:

$$C_1 = 0 \tag{32}$$

$$a = \frac{d}{K} - \alpha \Rightarrow K = \frac{\alpha}{2(\omega - \epsilon)} \tag{33}$$

$$\frac{b}{2} = \frac{e}{K} - b \Rightarrow \omega - \epsilon = \frac{\alpha}{2\bar{\lambda}}\left(1 + \sqrt{1 + \frac{2\alpha}{\lambda_a}\bar{\lambda}}\right), \quad \bar{\lambda} = \frac{4}{3}\frac{\lambda}{\lambda_a} \tag{34}$$

$$\frac{p}{3} = \frac{q}{K} - p \Rightarrow \lambda_m = \frac{4}{3}\lambda \tag{35}$$

Assuming this, the solution for u can now be found by integrating (30):

$$\int \frac{du}{u\sqrt{a + bu^2/2 + pu^4/3}} = \pm \int dz \tag{36}$$

To integrate the left-hand side, we consider two cases:

CASE I: Assume $p = 0$. It follows from (28) that either $\lambda = 0$ (no atom-molecular interaction) or $K = 0$ (no molecular component), which rules out this case. Note: If $K = 0$, $a > 0$, $b < 0$ and $\lambda \neq 0$, then we find that (36) degenerates to the one-soliton solution for u:

$$u(x,t) = 2\sqrt{\frac{2a}{|b|}}\,\text{sech}[\sqrt{a}(x - ct + D) \pm \phi] \tag{37}$$

with $e^{\pm\phi} = 2\frac{b}{|b|}\sqrt{|b|}$.

CASE II: Assume $p \neq 0$. In this case, we make the the substitution $v = Ku^2$, which transforms the integral on the left-hand side of (36) to

$$\int \frac{du}{u\sqrt{a + bu^2/2 + pu^3/3}} = \int \frac{dv}{v\sqrt{A + Bv + Cv^2}}, \tag{38}$$

where here $A = 4aK$, $B = 2b$ and $C = \frac{4p}{3K}$. This last integral depends on the sign of A, in particular.

$$\int \frac{dv}{v\sqrt{A+Bv+Cv^2}} = \begin{cases} -\frac{1}{2\sqrt{A}}\log\frac{2\sqrt{A(A+Bv+Cv^2)}+Bv+2A}{v} & (A>0) \\ -\frac{1}{2\sqrt{-A}}\sin^{-1}\left(\frac{Bv+2A}{v\sqrt{B^2-4AC}}\right) & (A<0) \\ -\frac{2\sqrt{Bv+Cv^2}}{Bv} & (A=0) \end{cases} \tag{39}$$

From this we find that bounded solutions for u and v exist only for $A > 0$ and $B^2 - 4AC > 0$. These conditions are satisfied when $2(k+\omega) > c^2$ and $\lambda < 0$, respectively, in which case

$$v(x,t) = \frac{4A^{3/2}}{Ae^{-\sqrt{A}(x-ct+C2)} + (B^2-4AC)e^{\sqrt{A}(x-ct+C2)} - 2\sqrt{A}B} \tag{40}$$

$$u(x,t) = \pm\sqrt{\frac{v(x,t)}{K}} \tag{41}$$

4. Small Amplitude Approximation

We now assume $|v| << |u|$. In this case, the αuv and $\lambda v^2 u$ terms can be dropped in the first MLE equation and the $\lambda_m v^3$ term can be dropped in the second equation to give

$$u_{zz} + (c^2 - 2\omega - 2k)u - 2\lambda_a u^3 = 0 \tag{42}$$

$$v_{zz} - 4(\lambda u^2 + 2k + \omega + \epsilon - c^2)v = 2\alpha u^2 \tag{43}$$

Integrating the first equation above yields

$$(u_z)^2 = au^2 + bu^4 + C_1 \tag{44}$$

where $a = 2(\omega + k) - c^2$ and $b = \lambda_a$. which we factor as

$$(u_z)^2 = (A - Bu^2)(C - Du^2) \tag{45}$$

Case I: Assume $C_1 = 0$ and again $a > 0$ and $b < 0$. Then

$$u(x,t) = K\text{sech}[L(x-ct) + C_2 \pm \phi] \tag{46}$$

where $K = 2\sqrt{\frac{a}{|b|}}$, $L = \sqrt{a}$, and $e^\phi = 2\sqrt{|b|}$.

Case II: Assume $C_1 \neq 0$. By requiring $AD \neq 0$, $BC \neq 0$ and making the substitutions $u = \sqrt{A/B}w$, $\kappa^2 = AD/BC$, it follows that the general solution for u can be expressed in terms of elliptic functions:

$$\frac{1}{\sqrt{BC}}\int \frac{dw}{\sqrt{(1-w^2)(1-\kappa^2 w^2)}} = \int dz \tag{47}$$

or

$$u(x,t) = \sqrt{\frac{A}{B}}\text{sn}(\sqrt{BC}(x-ct) + C_2, \kappa) \tag{48}$$

To find v, let us for the moment assume Case I holds so that

$$u(z) = K\text{sech}[L(z + C_2) \pm \phi], \tag{49}$$

where $K = 2\sqrt{a/b}$ and $L = \sqrt{a}$. Without loss of generality, we now assume $C_2 = 0$ and $\phi = 0$. Then (17) reduces to the nonhomogeneous Legendre differential equation

$$v_{zz} - 4[\lambda K^2\text{sech}^2(Lz) + \omega + \epsilon + 2k]v = \frac{\alpha}{2}K^2\text{sech}^2(Lz) \tag{50}$$

The substitutions $y = \tanh(Lz)$, $m^2 = 4(\omega + \epsilon + 2k)/L^2$, $n(n+1) = -4\lambda K^2/L^2$, and $r = 2\alpha K^2/L^2$ will put (50) into standard form:

$$[(1-y^2)v_y]_y - \frac{m^2}{1-y^2}v + n(n+1)v = r \tag{51}$$

Assuming m and n are non-negative integers, it follows that the homogeneous solutions of (51), denoted by $v_n^m(y)$, can be described in terms of Legendre polynomials. In particular, let $P_n(y)$ be the Legendre polynomial of degree n and $P_n^m(y)$ be its associated Legendre polynomial given by

$$P_n^m(y) = (-1)^m(1-y^2)^{m/2}\frac{d^m}{dy^m}P_n(y), \quad m = 0, 1, ..., n. \tag{52}$$

Then

$$v_n^m(y) = c_1 P_n^m(y) + c_2 Q_n^m(y) \tag{53}$$

where $Q_n^m(y)$ is the unbounded homogeneous solution to (51). Therefore, the nonhomogeneous solution becomes

$$v(y) = v_n^m(y) + R_n^m(y) \tag{54}$$

where $R_n^m(y)$ is a particular nonhomogeneous solution to (51).

Next, we consider certain special cases. For $m = 0$, we have

$$P_n^0(y) = P_n(y) \tag{55}$$

$$R_n^0(y) = \frac{r}{n^2+n} \tag{56}$$

For $n = 2$, we have

$$P_2^0(y) = \frac{1}{2}(3y^2-1) \tag{57}$$

$$P_2^1(y) = -3y(1-y^2)^{1/2} \tag{58}$$

$$P_2^2(y) = 3(1-y^2) \tag{59}$$

and

$$R_2^0(y) = \frac{r}{6} \tag{60}$$

$$R_2^1(y) = \frac{r}{3}(1-y^2) \tag{61}$$

$$R_2^2(y) = \frac{ry^2(3-2y^2)}{6(1-y^2)} \tag{62}$$

For $n = 4$, we have

$$R_4^0(z) = \frac{r}{20} \tag{63}$$

$$R_4^1(z) = \frac{r}{45}(1-z^2)(1+14z^2) \tag{64}$$

$$R_4^2(z) = \frac{rz^2(90-165z^2+77z^4)}{180(1-z^2)} \tag{65}$$

$$R_4^3(z) = \frac{r}{5}(1-z^2)(1-2z^2) \tag{66}$$

$$R_4^4(z) = -\frac{r(45-540z^2+930z^4-644z^6+161z^8)}{720(1-z^2)^2} \tag{67}$$

For $n = 6$, we have

$$R_6^0(z) = \frac{r}{42} \tag{68}$$

$$R_6^1(z) = \frac{r}{525}(19 - 127z^2 + 372z^4 - 264z^6) \tag{69}$$

$$R_6^2(z) = \frac{rz^2(210 - 770z^2 + 924z^4 - 363z^6)}{420(1 - z^2)} \tag{70}$$

$$R_6^3(z) = -\frac{r}{105}(1 - z^2)(1 - 68z^2 + 88z^4) \tag{71}$$

$$R_6^4(z) = \frac{r(105 + 6825z^2 - 25550z^4 + 36330z^6 - 23355z^8 + 5709z^{10})}{16800(1 - z^2)^2} \tag{72}$$

$$R_6^5(z) = \frac{r}{21}(1 - z^2)(3 - 12z^2 + 8z^4) \tag{73}$$

$$R_6^6(z) = -\frac{r(140 - 1890z^2 + 5250z^4 - 7210z^6 + 5445z^8 - 2178z^{10} + 363z^{12})}{2100(1 - z^2)^3} \tag{74}$$

More generally, we find for n even that

$$R_n^{2m+1}(z) = rS(z) \tag{75}$$

$$R_n^{2m}(z) = \frac{rT(z)}{(1 - z^2)^{m/2}} \tag{76}$$

where $S(z)$ and $T(z)$ are n-th and $(n + m)$-th degree polynomials, respectively.

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