SOLITON COLLISIONS AND GHOST PARTICLE RADIATION

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Abstract. We mathematically investigate particle collision for three-soliton solutions of the Korteweg-de Vries (KdV) equation and describe its corresponding ghost particle radiation. In particular, we prove that a collision between any two soliton particles results in an exchange of particle identities and an emission of a ghost particle pair. Moreover, collisions between any two ghost particles results in their fusion and is accompanied by fission of the corresponding anti-ghost particle. Mass and momentum are conserved for both soliton particle decay and ghost particle interaction at all times.

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1. Introduction

The subject of solitons as a nonlinear phenomenon is well established in the literature and much has been written about its asymptotic and particle-like properties (cf. [M], [DJ], [AC]). Yet the physical process of how solitons actually behave during collision has remained mysterious for the most part. Very little work has addressed the identity of Korteweg-de Vries (KdV) solitons as particles before and after collision. Historically, it was Zabusky and Kruskal [ZK] who originally proposed that solitons pass through each other cleanly with only a change in "phase shift", a viewpoint that was supported mathematically by P.D. Lax [La]. Two decades later, Bowtell and Stuart [BS] countered with a more consistent particle-like interpretation by proposing that solitons actually bounce off each other and exchange identities. Since KdV solitons must have distinct masses, the exchange of mass that occurs during collision was then explained as a continuous siphoning process, whereby the smaller soliton siphons mass from the larger soliton through the underlying baseline. This explanation is rather unnatural based on our physical understanding of particles. Others investigations have been made to understand collisions involving KdV solitons, but such works have failed to shed new light in resolving the debate (cf. citeHM, [Le], [MC]).

In [N], the author has presented what seems to be the correct argument for explaining two-soliton collisions. This was achieved by mathematically demonstrating that each soliton particle decays into two sub-particles upon collision. This break-up allows solitons to exchange identities and to emit a 'ghost' particle pair necessary to maintain conservation of mass and momentum. Such a solution was found by considering a particle-decomposition of each two-soliton solution of the Korteweg-de Vries equation in terms of eigenvalues. These eigenvalues are obtained from the corresponding soliton matrix defining the solution and isolate the decay of individual solitons. In particular, given two colliding solitons, e.g. those labeled as particles 1 and 2 in Figure 1, it was proven that each soliton splits into two upon collison. As a result there is an exchange of particle identities and an emission of a ghost particle pair as observed by particles 3 and 4 in Figure 1. Asymptotically, these ghost particles have the same sech-shaped profiles as solitons. Moreover, it was proven that conservation of mass and momentum holds in the decay of each soliton particle for all times. This demonstrates that our theory is consistent with the laws of classical mechanics.

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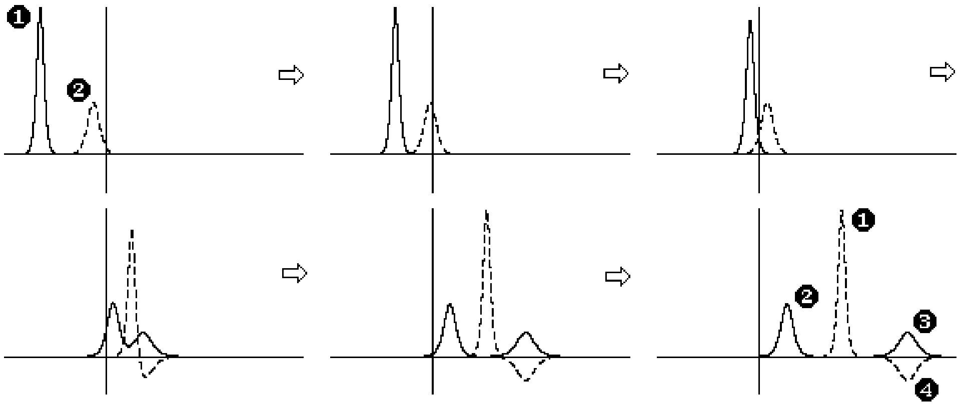


FIGURE 1. Two-Soliton Particle Collision

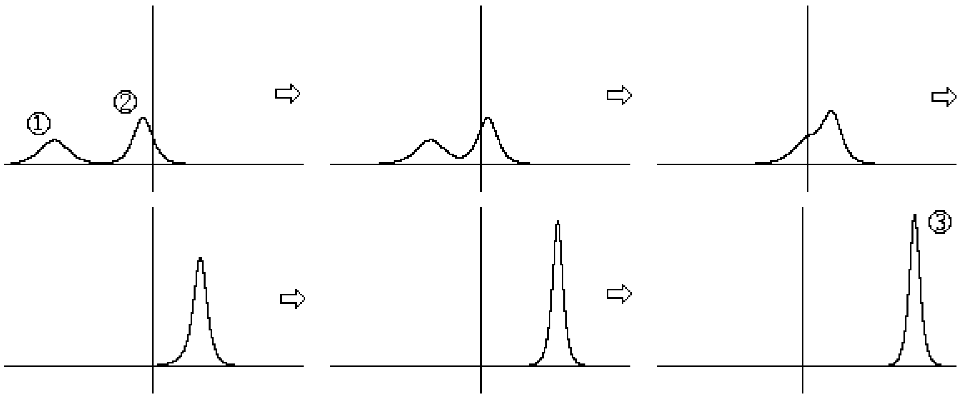


FIGURE 2. Two-Ghost Particle Collision

In this paper, we mathematically investigate particle decay for three-soliton solutions of the Korteweg-de Vries equation and describe the resulting interaction between ghost particles that are created in the process. Again, we find that each soliton collision results in the exchange of particle identities and the emission of a dual 'ghost' particle pair (cf. Theorem 3.1). As a novel feature, we discover that it is possible for ghost particles themselves to collide in which case they fuse to form a third ghost particle (cf. Theorem 4.2). Figure 2 illustrates two ghost particles, labeled as 1 and 2, that collide and fuse to become particle 3. Moreover, this fusion process is always complemented by fission of the corresponding anti-particle, i.e. the particle dual to particle 3, and obeys Richard Feynman's interpretation of anti-particles as particles moving backwards in time.

A duality therefore seems to exist between soliton interaction and ghost particle interaction. Our goal is to expose this duality by demonstrating that both interactions satisfy the same conservation laws and are equivalent in the sense that knowledge of one uniquely determines the other. Such an understanding of ghost particle radiation is important towards our broader understanding of solitons as nonlinear particles. Moreover, our theory of ghost particles is not far-fetched since it is now widely accepted that solitons can experience inelastic collisions and interact in a highly nontrivial manner. Indeed, certain kinds of optical vector solitons are known to split, fuse, and exchange energies during collision (cf. [AS], [KL], [KH])). Moreover, this type of behavior seems to be generic rather than exceptional for solitons. Since solitons have already found important applications in many areas of physics and engineering (cf. [AA], [HK], [Ra], [Re]), we expect these collision interactions to provide new applications in the future (cf. [JSS]).

Our paper is organized as follows. We begin by reviewing the eigenvalue-decomposition that is used to isolate the decay of solitons during collision. We then analysis the asymptotic behavior of these eigenvalues for three-solitons to establish mass exchange, ghost particle radiation, and various conservation laws. Lastly, we describe the fusion and fission of ghost particles for three-soliton collisions.

2. Preliminaries

2.1. N-**Solitons.** Let N be a positive integer and assume $u(x,t)$ to be an N-soliton solution of the KdV equation. The Miura transformation makes u become the potential function for the time-independent Schrodinger equation,

$$\psi_{xx} - [\lambda - u(x,0)]\psi = 0. \tag{1}$$

Since u is assumed to be reflectionless, the initial scattering data for $u(x,0)$ contains only a discrete energy spectrum. This means that λ takes on a discrete set of N negative energy eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_N\}$ with corresponding eigenfunctions $\{\psi_1, \psi_2, ..., \psi_N\}$. As standard we normalize these eigenfunctions and compute their normalized factors c_n commonly referred to as 'phase shifts':

$$\int_{-\infty}^{\infty} \psi_n^2 dx = 1, \qquad c_n = \lim_{x \to -\infty} e^{k_n x} \psi_n. \tag{2}$$

The initial scattering data is then used to produce the N-soliton formula for u (cf. [GGKM], [H], [WT]):

$$u(x,t) = -2 \frac{\partial^2}{\partial x^2} \log \det(I + A). \tag{3}$$

Here, the $N \times N$ *soliton matrix* A has entries defined by

$$A = (a_{mn}); \qquad a_{mn} = \frac{c_m(t) c_n(t)}{k_m + k_n} e^{(k_m + k_n)x}; \qquad c_n(t) = c_n e^{-4k_n^3 t}, \tag{4}$$

where the spectral parameter $k_n > 0$ is defined via the relation $\lambda_n = -k_n^2$.

2.2. **Soliton Particles.** We now turn to developing our working definition of a soliton particle. It is well known that A is symmetric and positive definite (cf. [KM],[GGKM],[WT]). This allows us to diagonalize it so that

$$U^{-1} A U = D \tag{5}$$

where $D = \text{diag}(\mu_1, ..., \mu_N)$ is a diagonal matrix, $\{\mu_1 > ... > \mu_N\}$ consists of the (ordered) set of real positive eigenvalues of A, and $U = (\mathbf{u}_1, ..., \mathbf{u}_N)$ is the orthogonal matrix consisting of an orthonormal basis of eigenvectors $\{\mathbf{u}_n\}$ of A. It follows that we can write $u(x,t)$ in terms of $\{\mu_n\}$ which we shall refer to as *decay eigenvalues* of u:

$$\begin{aligned}
u(x,t) &= -2\frac{\partial^2}{\partial x^2}\log\det(I+A) \\
&= -2\frac{\partial^2}{\partial x^2}\log\det[U^{-1}(I+A)U] \\
&= -2\frac{\partial^2}{\partial x^2}\log\det(I+D) \\
&= -2\frac{\partial^2}{\partial x^2}\log\prod_{n=1}^{N}[1+\mu_n(x,t)] \\
&= \sum_{n=1}^{N} -2\frac{\partial^2}{\partial x^2}\log[1+\mu_n(x,t)]. \qquad (6)
\end{aligned}$$

Definition 2.3. We define

$$s_n(\nu_n) \equiv -2k_n^2 \operatorname{sech}^2(k_n\nu_n), \qquad n=1,...,N, \qquad (7)$$

to be the n-th *soliton particle* of u where $\nu_n = x - 4k_n^2 t$. Moreover, we shall refer to

$$u_n(x,t) \equiv -2\frac{\partial^2}{\partial x^2}\log[1+\mu_n(x,t)] \qquad (8)$$

as the *decay function* of s_n and to the sum $u = \sum_{n=1}^{N} u_n$ as derived in (6) as the *decay decomposition* of u.

Next, we introduce our notion of a 'ghost' particle which is needed to explain our theory of soliton interaction.

Definition 2.4. Assume m and n are positive integers with $m < n$. We define

$$G_{mn} = \begin{pmatrix} \frac{c_m^2}{2k_m} e^{2k_m\nu_{mn}} & \frac{c_m c_n}{k_m+k_n} e^{(k_m+k_n)\nu_{mn}} \\ \frac{c_m c_n}{k_m+k_n} e^{(k_m+k_n)\nu_{mn}} & \frac{c_n^2}{2k_n} e^{2k_n\nu_{mn}} \end{pmatrix} \qquad (9)$$

to be the 2×2 *ghost submatrix* of A corresponding to the pair $\{m,n\}$, where $\nu_{mn} = \nu_{nm} = x - 4k_{mn}^2 t$ and $k_{mn} = k_m^2 + k_m k_n + k_n^2$. In addition, denote by γ_{mn} and γ_{nm} to be the eigenvalues of G_{mn} and order them so that $\gamma_{nm} > \gamma_{mn}$.

Next, define $\delta_{mn} = \delta_{nm}$ via the relation $e^{2(k_m-k_n)\delta_{mn}} = \frac{c_m^2 k_n}{c_n^2 k_m}$. Then we shall refer to

$$g_{nm}(\nu_{nm}+\delta_{nm}) = -2\frac{\partial^2}{\partial \nu_{nm}^2}\log\gamma_{nm} \qquad (10)$$

as the *ghost particle* corresponding to the pair $\{m,n\}$ and

$$g_{mn}(\nu_{mn}+\delta_{mn}) = -2\frac{\partial^2}{\partial \nu_{mn}^2}\log\gamma_{mn} \qquad (11)$$

as its *anti-ghost particle*.

Theorem 2.5. *([N], Theorem 3.6) The particles g_{nm} and g_{mn} enjoy the following properties:*

(i) $g_{nm}(\nu_{nm}) = 8k_1k_2 \dfrac{\cosh[(k_1-k_2)\nu_{nm}]}{\left[\frac{(k_1+k_2)^2}{(k_1-k_2)^2}\cosh^2[(k_1-k_2)\nu_{nm}]-1\right]^{3/2}}.$

(ii) $g_{mn}(\nu_{mn}+\delta_{mn}) = -g_{nm}(\nu_{nm}+\delta_{nm}).$

(iii) $\displaystyle\int_{-\infty}^{\infty} g_{mn}(\nu_{mn})d\nu_{mn} = 4(k_m-k_n).$

(iv) $g_{mn}(\nu_{mn}+\delta_{mn}) \sim O(\operatorname{sech}^2[(k_m-k_n)(\nu_{mn}+\delta_{mn})])$ *as* $\nu_{mn}\to\pm\infty$.

(v) $|g_{mn}| \le \dfrac{(k_m-k_n)^2(k_m+k_n)}{\sqrt{k_m k_n}}$ *with equality holding precisely when* $\nu_{mn}=-\delta_{mn}$.

Remark 2.6. We remark that Theorem 2.5 shows the ghost particle g_{nm} can be viewed as a nonlinear difference between the soliton particles s_m and s_n as defined by (7).

2.7. **Asymptotic Matrices.** Since we shall be investigating the asymptotic behavior of the eigenvalues of soliton matrices, the following notion of *asmptotic* matrices will be useful to us.

Definition 2.8. Let $A(x,t)$ and $B(x,t)$ be $N\times N$ matrices in the variables x and t and $C(x)$ an $N\times N$ matrix depending on x only. We shall say that A *converges uniformly in* x *on arbitrary compact subsets of* $\mathbb{R}$, or simply *converges*, to C as $t\to\pm\infty$, and write $A\xrightarrow{\text{u}} C$, to mean

$$\lim_{t\to\pm\infty} A(x,t) = C(x) \tag{12}$$

uniformly in x on arbitrary compact subsets of $\mathbb{R}$ with respect to the Euclidean norm, i.e. $\| A(x,t)-C(x) \|\to 0$, where $\|\cdot\|$ is defined by

$$\| A \|^2 = \sum_{m,n}^{N} |a_{mn}|^2. \tag{13}$$

Moreover, we shall say that A is *asymptotic* to B as $t\to\pm\infty$, and write $A\sim B$, to mean $A(x,t)-B(x,t)\xrightarrow{\text{u}} 0$ (the zero matrix) as $t\to\pm\infty$.

The following standard result will be useful to us.

Lemma 2.9. *(*[MM]*, III.3.5.10, p. 163) Let* $A(x,t)$, $B(x,t)$ *and* $C(x)$ *be* $N\times N$ *normal matrices with corresponding sets of eigenvalues* $\{\alpha_1(x,t),...,\alpha_N(x,t)\}$, $\{\beta_1(x,t),...,\beta_N(x,t)\}$ *and* $\{\gamma_1(x),...,\gamma_N(x)\}$, *respectively.*

(i) If $A\xrightarrow{u} C$ *as* $t\to\pm\infty$, *then there exists a permutation* $\sigma\in S_N$ *of* $\{1,...,N\}$ *such that* $\alpha_n\xrightarrow{u}\gamma_{\sigma(n)}$ *as* $t\to\pm\infty$ *for every* $n=1,...,N$.

(ii) If $A\sim B$ *as* $t\to\pm\infty$, *then there exists a permutation* $\sigma\in S_N$ *of* $\{1,...,N\}$ *such that* $\alpha_n-\beta_{\sigma n}\xrightarrow{u} 0$ *as* $t\to\pm\infty$ *for every* $n=1,...,N$.

3. Particle Decay of Three-Solitons

In this section we assume $N=3$ and investigate the asymptotic behavior of the decay functions u_1, u_2 and u_3 as a means of understanding soliton interaction. We are now ready to present our theorem describing particle decay of three-solitons.

Theorem 3.1. *(*$N=3$*) The following asymptotic relations hold for* u_1, u_2, *and* u_3*:*

(i) $u_1: \quad s_1^-\to s_3^+ + g_{32}^+ + g_{21}^+$,

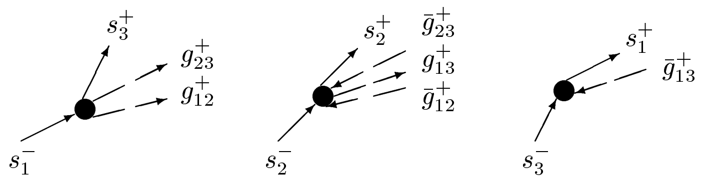
(ii) $u_2: \quad s_2^-\to s_2^+ + g_{23}^+ + g_{31}^+ + g_{12}^+$,

(iii) $u_3: \quad s_3^-\to s_1^+ + g_{13}^+$,

where in the notation above $s_n^\pm = s_n(\nu_n+\Delta_n^\pm)$, $g_{mn}^\pm = g_{mn}(\nu_{mn}+\Delta_{mn}^\pm)$, *and each asymptotic equation indicates behavior for* $t\to-\infty$ *and for* $t\to\infty$. *For* u_1 *say, this would mean*

$$\lim_{\substack{\nu_1 \text{ fixed}\\ t\to-\infty}} u_1 = s_1^-, \qquad \lim_{\substack{\nu_2 \text{ fixed}\\ t\to\infty}} u_1 = s_3^+, \tag{14a}$$

$$\lim_{\substack{\nu_{12} \text{ fixed}\\ t\to\infty}} u_1 = g_{12}^+, \qquad \lim_{\substack{\nu_{23} \text{ fixed}\\ t\to\infty}} u_1 = g_{23}^+. \tag{14b}$$



(a) Diagram of u_1 (b) Diagram of u_2 (c) Diagram of u_3

FIGURE 3. Space-time plots of three-soliton decay: $u = u_1 + u_2 + u_3$

Moreover, the relative phase shifts Δ_n^-, Δ_n^+, and $\Delta_{mn}^+ = \Delta_{nm}^+$ are defined by (assuming $m < n$)

$$e^{2k_n\Delta_n^-} = \frac{c_n^2}{2k_n}\prod_{i=1}^{n-1}\frac{(k_i-k_n)^2}{(k_i+k_n)^2}, \tag{15a}$$

$$e^{2k_n\Delta_n^+} = \frac{c_n^2}{2k_n}\prod_{i=n+1}^{N}\frac{(k_n-k_i)^2}{(k_n+k_i)^2}, \tag{15b}$$

$$e^{2(k_m-k_n)\Delta_{mn}^+} = \frac{\dfrac{c_m^2}{2k_m}\displaystyle\prod_{i=m+1}^{n-1}\frac{(k_m-k_i)^2}{(k_m+k_i)^2}}{\dfrac{c_n^2}{2k_n}\displaystyle\prod_{i=m+1}^{n-1}\frac{(k_i-k_n)^2}{(k_i+k_n)^2}}. \tag{15c}$$

Space-time plots corresponding to the asymptotic behavior described in Theorem 3.1 are drawn in Figure 3. We note that this asymptotic behavior is independent of the order of soliton collisions, i.e. independent of whether s_1 collides with s_2 first or s_2 collides with s_3 first.

Proof of Theorem 3.1. Our approach to proving (i), (ii), and (iii) basically involves analyzing each decay function from the perspective of its relevant moving frames:

Proof of (i): We divide the argument into cases by treating each of the four moving frames $\{\nu_1, \nu_2, \nu_{21}, \nu_{32}\}$ as a separate case:

CASE I: Assume ν_1 is fixed. Then it follows from (50) that

$$A(\nu_1, t) \xrightarrow{\text{u}} D \equiv \frac{c_1^2}{2k_1}e^{2k_1\nu_1}E_{11} \tag{16}$$

as $t \to -\infty$ where E_{11} is the 3×3 elementary matrix with 1 in its $(1,1)$-entry and 0's everywhere else. This is because the entries of $A - D$ consist of decaying exponential terms. Since A and D are Hermitian, we can apply Lemma 2.9 to obtain an ordering $\sigma \in S_3$ of the eigenvalues $\{d_1 = \frac{c_1^2}{2k_1}e^{2k_1\nu_1}, d_2 = 0, d_3 = 0\}$ of D so that $\mu_n \xrightarrow{\text{u}} d_{\sigma(n)}$ as $t \to -\infty$. In fact, σ will always be the identity element by our ordering of the eigenvalues. In particular, this implies that $\mu_1 \xrightarrow{\mathrm{u}} d_1$ as $t \to -\infty$ and hence

$$\begin{aligned}\lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} u_1 &= \lim_{\substack{\nu_1 \text{ fixed} \\ t\to-\infty}} -2\frac{\partial^2}{\partial x^2}\log(1+\mu_1) \\ &= -2\frac{\partial^2}{\partial x^2}\log(1+\frac{c_1^2}{2k_1}e^{2k_1\nu_1}) \\ &= s_1(\nu_1+\Delta_1^-),\end{aligned}$$

where we have used uniform convergence in x to pass the limit through the differentiation.

CASE II: Assume ν_3 is fixed. Applying an argument similar to that used in CASE I, we find that

$$\lim_{\substack{\nu_3 \text{ fixed} \\ t\to\infty}} u_1 = s_3(\nu_3+\Delta_3^+).$$

CASE III: Assume ν_{21} is fixed. In this case, it is more appropriate to determine the asymptotic behavior of u_1 as $t\to\infty$ by working with the matrix A_{12} as defined by (53) and its eigenvalue μ_1^{12} because of (53). It follows from part (ii) of Lemma A.1 that

$$A_{12}(\nu_{12},t) \xrightarrow{\mathrm{u}} \begin{pmatrix} G_{12} & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

as $t\to\infty$. Here, G_{12} is the 2×2 ghost principal submatrix of A defined by (53). Since G_{12} has eigenvalues γ_{12} and γ_{21} with $\gamma_{21}>\gamma_{12}$, Lemma 2.9 says that $\mu_1^{21} \xrightarrow{\mathrm{u}} \gamma_{21}$ as $t\to\infty$. We then conclude from (57) that

$$\begin{aligned}\lim_{\substack{\nu_{21} \text{ fixed} \\ t\to\infty}} u_1 &= \lim_{\substack{\nu_{21} \text{ fixed} \\ t\to\infty}} -2\frac{\partial^2}{\partial x^2}\log(1+\mu_1) \\ &= \lim_{\substack{\nu_{21} \text{ fixed} \\ t\to\infty}} -2\frac{\partial^2}{\partial x^2}\log\left(e^{-8(k_p^2k_q+k_pk_q^2)t}+\mu_1^{21}\right) \\ &= -2\frac{\partial^2}{\partial x^2}\log\gamma_{21} \\ &= g_{21}(\nu_{21}+\Delta_{21}^+),\end{aligned}$$

as desired.

CASE IV: Assume ν_{32} is fixed. We shall let the reader check that an argument analogous to CASE III (with ν_{21} fixed) can be applied to prove that

$$\lim_{\substack{\nu_{32} \text{ fixed} \\ t\to\infty}} u_1 = g_{32}(\nu_{32}+\Delta_{32}^+).$$

This completes the proof of part (i).

Proof of (ii): We follow (i) by applying a similar analysis to u_2 and consider four separate cases corresponding to the four relevant moving frames ν_2, ν_{23}, ν_{31}, and ν_{12}:

CASE I: Assume ν_2 is fixed. We first consider the situation where $t\to-\infty$. Now, depending on whether s_1 and s_2 collide first or s_2 and s_3 collide first, we have that either

$$k_1(k_2^2-k_1^2)+k_3(k_2^2-k_3^2)>0 \text{ or } k_1(k_2^2-k_1^2)+k_3(k_2^2-k_3^2)\le 0. \tag{18}$$

If $k_1(k_2^2-k_1^2)+k_3(k_2^2-k_3^2)>0$, then it follows from (50) that A is asymptotic to

$$\tilde{A}=\begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1\nu_2+8k_1(k_2^2-k_1^2)t} & \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_2+4k_1(k_2^2-k_1^2)t} & 0 \\ \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_2+4k_1(k_2^2-k_1^2)t} & \frac{c_2^2}{2k_2}e^{2k_2\nu_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

as $t\to-\infty$. Since $\tilde{A}$ diverges as $t\to-\infty$, we shall work with its inverse instead. Using the fact that $\tilde{A}$ contains the 2×2 ghost principal submatrix G_{12}, we find that

$$\tilde{A}^{-1}\xrightarrow{\mathrm{u}}\frac{(k_1+k_2)^2}{(k_1-k_2)^2}\frac{2k_2}{c_2^2}e^{-2k_2\nu_2}E_{22} \tag{20}$$

as $t\to-\infty$. As μ_2 is defined to be the second largest eigenvalue of A, this implies

$$\mu_2^{-1}(\nu_2,t)\xrightarrow{\mathrm{u}}\frac{(k_1+k_2)^2}{(k_1-k_2)^2}\frac{2k_2}{c_2^2}e^{-2k_2\nu_2} \tag{21}$$

or equivalently,

$$\mu_2(\nu_2,t)\xrightarrow{\mathrm{u}}\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2} \tag{22}$$

as $t\to-\infty$. On the other hand, if $k_1(k_2^2-k_1^2)+k_3(k_2^2-k_3^2)\le 0$, then it follows again from (50) that A is asymptotic to

$$\tilde{A}=\begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1\nu_2+8k_1(k_2^2-k_1^2)t} & \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_2+4k_1(k_2^2-k_1^2)t} & \tilde{a}_{13} \\ \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)\nu_2+4k_1(k_2^2-k_1^2)t} & \frac{c_2^2}{2k_2}e^{2k_2\nu_2} & 0 \\ \tilde{a}_{13} & 0 & 0 \end{pmatrix} \tag{23}$$

where $\tilde{a}_{13}=\frac{c_1c_3}{k_1+k_3}e^{(k_1+k_3)\nu_2+4k_1(k_2^2-k_1^2)t+4k_3(k_2^2-k_3^2)t}$ as $t\to-\infty$. By computing the inverse of $\tilde{A}$, we find that (20) again holds and therefore (22) holds. Hence,

$$\begin{aligned}\lim_{\substack{\nu_1 \text{ fixed}\\ t\to-\infty}} u_2 &= \lim_{\substack{\nu_1 \text{ fixed}\\ t\to-\infty}} -2\frac{\partial^2}{\partial x^2}\log(1+\mu_2)\\ &= -2\frac{\partial^2}{\partial x^2}\log\left(1+\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2}\right)\\ &= s_2(\nu_2+\Delta_2^-).\end{aligned}$$

Let us now consider the situation where $t\to\infty$. By copying the argument used for $t\to-\infty$, we find that

$$\mu_2(\nu_2,t)\xrightarrow{\mathrm{u}}\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2}. \tag{24}$$

as $t\to\infty$. Hence,

$$\begin{aligned}\lim_{\substack{\nu_1 \text{ fixed}\\ t\to\infty}} u_2 &= \lim_{\substack{\nu_1 \text{ fixed}\\ t\to\infty}} -2\frac{\partial^2}{\partial x^2}\log(1+\mu_2)\\ &= -2\frac{\partial^2}{\partial x^2}\log\left(1+\frac{(k_2-k_3)^2}{(k_2+k_3)^2}\frac{c_2^2}{2k_2}e^{2k_2\nu_2}\right)\\ &= s_2(\nu_2+\Delta_2^+).\end{aligned}$$

CASE II: Assume ν_{12} is fixed. It follows from the results of (i), CASE III, that $\mu_2^{12} \xrightarrow{\text{u}} \gamma_{12}$ as $t \to \infty$. Hence,

$$\begin{aligned}
\lim_{\substack{\nu_{12}\ \text{fixed} \\ t\to\infty}} u_2 &= \lim_{\substack{\nu_{12}\ \text{fixed} \\ t\to\infty}} 2\frac{\partial^2}{\partial x^2}\log(1+\mu_2) \\
&= \lim_{\substack{\nu_{12}\ \text{fixed} \\ t\to\infty}} -2\frac{\partial^2}{\partial x^2}\log\left(e^{-8(k_1^2k_2+k_1k_2^2)t}+\mu_2^{12}\right) \\
&= -2\frac{\partial^2}{\partial x^2}\log\gamma_{12} \\
&= g_{12}(\nu_{12}+\Delta_{12}^+).
\end{aligned}$$

CASE III: Assume ν_{31} is fixed. Again, it will be more appropriate to work with the matrix A_{13} as defined by (53) and its eigenvalue μ_2^{13} because of (57). However, since A_{13} diverges in this case, we shall work with its inverse instead. It then follows from (iv) of Lemma B.1 that

$$A_{13}^{-1}(\nu_{13},t) \xrightarrow{\text{u}} \tilde{A}_{13}^{-1} = \begin{pmatrix} a^{11}\frac{2k_1}{c_1^2}e^{2k_1\nu_{13}} & 0 & a^{13}\frac{k_1+k_3}{c_1c_3}e^{(k_1+k_3)\nu_{13}} \\ 0 & 0 & 0 \\ a^{13}\frac{k_1+k_3}{c_1c_3}e^{(k_1+k_3)\nu_{13}} & 0 & a^{33}\frac{2k_3}{c_3^2}e^{2k_3\nu_{13}} \end{pmatrix} \tag{25}$$

as $t \to \infty$ where

$$\tilde{A}_{13} = \begin{pmatrix} \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\frac{c_1^2}{2k_1}e^{2k_1\nu_{13}} & 0 & \frac{(k_1-k_2)(k_2-k_3)}{(k_1+k_2)(k_2+k_3)}\frac{c_1c_3}{k_1+k_3}e^{(k_1+k_3)\nu_{13}} \\ 0 & 0 & 0 \\ \frac{(k_1-k_2)(k_2-k_3)}{(k_1+k_2)(k_2+k_3)}\frac{c_1c_3}{k_1+k_3}e^{(k_1+k_3)\nu_{13}} & 0 & \frac{(k_2-k_3)^2}{(k_2+k_3)^2}\frac{c_3^2}{2k_3}e^{2k_3\nu_{13}} \end{pmatrix}. \tag{26}$$

Now, denote by $\tilde{\mu}_{13}$ and $\tilde{\mu}_{31}$ to be the two positive eigenvalues of $\tilde{A}_{13}$ with $\tilde{\mu}_{31} > \tilde{\mu}_{13}$. Then it follows that μ_2^{31}, being the second largest eigenvalue of A_{13}, converges uniformly to the largest eigenvalue of $\tilde{A}_{13}$, i.e. $\mu_2^{31}(\nu_{31},t) \xrightarrow{\text{u}} \tilde{\mu}_{31}(\nu_{31})$, as $t \to \infty$. Hence,

$$\begin{aligned}
\lim_{\substack{\nu_{31}\ \text{fixed} \\ t\to\infty}} u_2 &= \lim_{\substack{\nu_{31}\ \text{fixed} \\ t\to\infty}} 2\frac{\partial^2}{\partial x^2}\log(1+\mu_2) \\
&= \lim_{\substack{\nu_{31}\ \text{fixed} \\ t\to\infty}} -2\frac{\partial^2}{\partial x^2}\log\left(e^{-8(k_1^2k_3+k_1k_3^2)t}+\mu_2^{31}\right) \\
&= -2\frac{\partial^2}{\partial x^2}\log\tilde{\mu}_{31} \\
&= g_{31}(\nu_{31}+\Delta_{31}^+),
\end{aligned}$$

where we have made use of the fact that $\tilde{A}_{13}$ has a 2×2 ghost principal submatrix of the form

$$G_{13} = \begin{pmatrix} \frac{\tilde{c}_1^2}{2k_1}e^{2k_1\nu_{13}} & \frac{\tilde{c}_1\tilde{c}_3}{k_1+k_3}e^{(k_1+k_3)\nu_{13}} \\ \frac{\tilde{c}_1\tilde{c}_3}{k_1+k_3}e^{(k_1+k_3)\nu_{13}} & \frac{\tilde{c}_3^2}{2k_3}e^{2k_3\nu_{13}} \end{pmatrix} \tag{27}$$

with normalization constants

$$\tilde{c}_1 = c_1\left(\frac{k_1-k_2}{k_1+k_2}\right), \tilde{c}_3 = c_3\left(\frac{k_2-k_3}{k_2+k_3}\right), \tag{28}$$

CASE IV: Assume ν_{23} is fixed. The argument here is similar to that of (i), CASE III, and will be omitted. This completes the proof of (ii).

Proof of (iii): Since the proof here follows from the results of (i) and (ii) and uses the same line of argument as employed above, we shall end the proof of our theorem here. □

Using equations (15a)-(15c), it is easy to verify that each decay functions in Theorem 3.1 conserve its total phase shift as described below:

Corollary 3.2. *Conservation of phase shift ($N = 3$):*

(i) $k_1\Delta_1^- = k_3\Delta_3^+ - (k_3 - k_2)\Delta_{32}^+ - (k_2 - k_1)\Delta_{21}^+,$

(ii) $k_2\Delta_2^- = k_2\Delta_2^+ - (k_2 - k_3)\Delta_{23}^+ - (k_3 - k_1)\Delta_{31}^+ - (k_1 - k_2)\Delta_{12}^+,$

(iii) $k_3\Delta_3^- = k_1\Delta_1^+ - (k_1 - k_3)\Delta_{13}^+.$

We next prove that each decay function conserves its mass and momentum.

Theorem 3.3. *(i) Conservation of mass ($N = 3$):*

$$\int_{-\infty}^{\infty} u_n(x,t)dx = -4k_n, \qquad n = 1, ..., N.$$

(ii) Conservation of momentum ($N = 3$):

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu_n(x,t)dx = -16k_n^3, \qquad n = 1, ..., N.$$

Proof. (i) We directly calculate the integral

$$\begin{aligned} \int_{-\infty}^{\infty} u_n(x,t)dx &= \int_{-\infty}^{\infty}\left[-2\tfrac{\partial^2}{\partial x^2}\log(1+\mu_n)\right]dx \\ &= \left[-2\tfrac{\partial}{\partial x}\log(1+\mu_n)\right]_{-\infty}^{\infty} \\ &= -2\left[\tfrac{\mu_n'}{1+\mu_n}\right]_{-\infty}^{\infty} \\ &= -4k_n. \end{aligned}$$

Here, we have used the fact that $\mu_n', \mu_n \to 0$ as $x \to -\infty$ and $\mu_n/e^{2k_n x} \to e^{-8k_n^3 t + 2k_n\Delta_n^+}$ and $\mu_n/e^{2k_n x} \to 2k_n e^{-8k_n^3 t + 2k_n\Delta_n^+}$ as $x \to \infty$ (cf. Appendix, Lemma C.1).

(ii) Integration by parts yields

$$\begin{aligned} \int_{-\infty}^{L} xu_n(x,t)dx &= \left[-2x\tfrac{\partial}{\partial x}\log(1+\mu_n)\right]_{-\infty}^{L} - \int_{-\infty}^{L}\left[-2\tfrac{\partial}{\partial x}\log(1+\mu_n)\right]dx \\ &= \left[-2x\tfrac{\partial}{\partial x}\log(1+\mu_n)\right]_{-\infty}^{L} + 2\left[\log(1+\mu_n)\right]_{-\infty}^{L} \\ &= -2L\tfrac{\mu_n'(L,t)}{(1+\mu_n(L,t))} + 2\log(1+\mu_n(L,t)) \\ &\sim -4k_nL + 4k_n(L - 4k_n^2 t + \Delta_n^+) \end{aligned}$$

as $L \to \infty$. It follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu_n(x,t)dx = -16k_n^3, \qquad n = 1, 2,$$

as desired. □

It follows immediately from Theorem 3.3 that

Corollary 3.4. *If the center of mass of u_n is defined to be*

$$x_n(t) \equiv \frac{\int_{-\infty}^{\infty} xu_n(x,t)dx}{\int_{-\infty}^{\infty} u_n(x,t)dx}, \tag{29}$$

then $x_n(t)$ moves with constant velocity $4k_n^2$, i.e.

$$\frac{dx_n}{dt} = 4k_n^2, \qquad n = 1, ..., 3. \tag{30}$$

4. Ghost Particle Interaction

In this section, we investigate the interaction of multiple ghost particles that may appear in each decay function. Our approach is to decompose each decay function u_n into its 'soliton' and 'ghost' components:

$$u_n = -2\frac{\partial^2}{\partial x^2}\log(1+\mu_n) \tag{31}$$

$$= -2\left[\frac{(1+\mu_n)\mu_n'' - (\mu_n')^2}{(1+\mu_n)^2}\right] \tag{32}$$

$$= -2\left[\frac{\mu_n''}{(1+\mu_n)^2}\right] - 2\left[\frac{\mu_n\mu_n'' - (\mu_n')^2}{\mu_n^2}\right]\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{33}$$

$$= -2\left[\frac{\mu_n''}{(1+\mu_n)^2}\right] - 2\left(\frac{\partial}{\partial x^2}\log\mu_n\right)\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{34}$$

$$= u_n^s + u_n^g\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{35}$$

Definition 4.1. We shall refer to

$$u_n^s = -2\left[\frac{\mu_n''}{(1+\mu_n)^2}\right] \tag{36}$$

as the *soliton component* of u_n,

$$u_n^g = -2\frac{\partial}{\partial x^2}\log\mu_n \tag{37}$$

as the *ghost function* of u_n, and

$$u_n^g\left(\frac{\mu_n}{1+\mu_n}\right)^2 \tag{38}$$

as the *ghost component* of u_n.

The following theorem tells us that the ghost functions defined by (37) in essence describe ghost particle interaction.

Theorem 4.2. *($N=3$) The decay of each ghost function u_n^g is described as follows:*

(i) $u_1^g : g_{31}^- \to g_{32}^+ + g_{21}^+$,

(ii) $u_2^g : g_{21}^- + g_{13}^- + g_{32}^- \to g_{23}^+ + g_{31}^+ + g_{12}^+$,

(iii) $u_3^g : g_{12}^- + g_{23}^- \to g_{13}^+$,

where in the notation above $g_{mn}^\pm \equiv g_{mn}(\nu_{mn} + \Delta_{mn}^\pm)$, $\Delta_{mn}^+ = \Delta_{nm}^+$ is given by (15c), and $\Delta_{mn}^- = \Delta_{nm}^+$ is given by (assuming $m<n$)

$$e^{2(k_m-k_n)\Delta_{mn}^-} = \frac{\dfrac{c_m^2}{2k_m}\displaystyle\prod_{i=1}^{m-1}\frac{(k_i+k_m)^2}{(k_i-k_m)^2}\prod_{i=n}^{N}\frac{(k_m+k_i)^2}{(k_m-k_i)^2}}{\dfrac{c_n^2}{2k_n}\displaystyle\prod_{i=1}^{m}\frac{(k_i+k_n)^2}{(k_i-k_n)^2}\prod_{i=n+1}^{N}\frac{(k_n+k_i)^2}{(k_n-k_i)^2}}. \tag{39}$$

Proof. By (57), the asymptotic behavior of u_n^g for $n = 1,2,3$ and $t\to\infty$ with respect to a ghost moving frame ν_{pq} is exactly the same as the asymptotic behavior of u_n established in the proof of Theorem 3.1, i.e.

$$\lim_{\substack{\nu_{pq}\ \text{fixed}\\ t\to\infty}} u_n^g = \lim_{\substack{\nu_{pq}\ \text{fixed}\\ t\to\infty}} u_n. \tag{40}$$

Therefore, it remains to verify their behavior for $t \to -\infty$. This however is straightforward and as the following lemma demonstrates the behavior of u_n for $t \to -\infty$ is reflected in the behavior of $-u_{N-n+1}$ for $t \to \infty$ modulo phase shift, i.e. subject to a change in the normalization constants of the soliton matrix $A(x,t)$. This completes the proof. □

Remark 4.3. Theorem 4.2 reveals then that two colliding ghost particles, say g_{12}^- and g_{23}^- appearing in u_3^g, will fuse into a third ghost particle, g_{13}^+. By duality, we have that anti-particle g_{13}^- appearing in u_1^g will split into g_{12}^+ and g_{23}^+. Both fusion and fission occurs in u_2^g.

Lemma 4.4. *Let $A(x,t)$ and $\tilde{A}(x,t)$ be soliton matrices with normalization constants $\{c_n\}$ and $\{\tilde{c}_n\}$, respectively, that are related by*

$$\tilde{c}_n = c_n \prod_{\substack{i<j \\ i=n \text{ or } j=n}}^{N} \frac{k_i + k_j}{k_i - k_j}. \tag{41}$$

Moreover, let $A(x,t)$ and $\tilde{A}(x,t)$ have eigenvalues $\{\mu_n(x,t)\}$ and $\{\tilde{\mu}_n(x,t)\}$, respectively. Then

$$\mu_n(-x,-t) = \frac{1}{\tilde{\mu}_{N-n+1}(x,t)}. \tag{42}$$

Proof. Using Lemma B.1, part (iii) of the Appendix, we find that the relation

$$A(-x,-t) = \tilde{A}^{-1}(x,t) \tag{43}$$

is valid from which our result follows immediately. □

As with soliton particle decay, we now demonstrate that ghost particle decay satisfies the same conservation laws.

Corollary 4.5. *Conservation of ghost phase shift ($N=3$):*

(i) $(k_3 - k_1)\Delta_{31}^- = (k_3 - k_2)\Delta_{32}^+ + (k_2 - k_1)\Delta_{21}^+$, *(ii)* $(k_2 - k_1)\Delta_{21}^- + (k_1 - k_3)\Delta_{13}^- + (k_3 - k_2)\Delta_{32}^- = (k_2 - k_3)\Delta_{23}^+ + (k_3 - k_1)\Delta_{31}^+ + (k_1 - k_2)\Delta_{12}^+$, *(iii)* $(k_1 - k_2)\Delta_{12}^- + (k_2 - k_3)\Delta_{23}^- = (k_1 - k_3)\Delta_{13}^+$.

Theorem 4.6. *(i) Conservation of ghost mass ($N=3$):*

$$\int_{-\infty}^{\infty} u_n^g(x,t)dx = -4(k_n - k_{N-n+1}), \qquad n = 1,2,3. \tag{44}$$

(ii) Conservation of ghost momentum ($N=3$):

$$\frac{d}{dt}\int_{-\infty}^{\infty} x u_n^g(x,t)dx = -16(k_n^3 - k_{N-n+1}^3), \qquad n = 1,2,3. \tag{45}$$

Proof. (i) We directly integrate to obtain

$$\begin{aligned}
\textstyle\int_{-\infty}^{\infty} u_n^g(x,t)dx &= \textstyle\int_{-\infty}^{\infty} \left[-2\frac{\partial^2}{\partial x^2}\log(\mu_n)\right] dx \\
&= \left[-2\frac{\partial}{\partial x}\log(\mu_n)\right]_{-\infty}^{\infty} \\
&= -2\left[\frac{\mu_n'}{\mu_n}\right]_{-\infty}^{\infty} \\
&= -4(k_n - k_{N-n+1}).
\end{aligned}$$

Again, we have used the fact that $\mu_n'/e^{2k_n x} \to c_n^2(t)$, $\mu_n/e^{2k_n x} \to c_n^2(t)/(2k_n)$ as $x \to \infty$ (cf. Lemma C.1) and $\mu_n(-x,-t) = 1/\tilde{\mu}_n(x,t)$ as $x \to \infty$ (cf. Lemma 4.4).

(ii) It suffices to integrate by parts and use the asymptotic relations described in (i) to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} xu_n^g(x,t)dx &= \left[-2x\tfrac{\partial}{\partial x}\log\mu_n\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left[-2\tfrac{\partial}{\partial x}\log\mu_n\right]dx \\ &= -2\left[x\tfrac{\mu_n'(x,t)}{\mu_n(x,t)} - \log(\mu_n(x,t))\right]_{-\infty}^{\infty} \\ &= -16(k_n^3 - k_{N-n+1}^3),\end{aligned}$$

as desired. □

Corollary 4.7. *If the center of mass of u_n^g is defined to be*

$$x_n^g(t) \equiv \frac{\int_{-\infty}^{\infty} xu_n^g(x,t)dx}{\int_{-\infty}^{\infty} u_n^g(x,t)dx}, \tag{46}$$

then $x_n^g(t)$ moves with constant velocity $4(k_n^2 + 2k_nk_{N-n+1} + k_{N-n+1}^2)$, i.e.

$$\frac{dx_n}{dt} = 4(k_n^2 + 2k_nk_{N-n+1} + k_{N-n+1}^2), \qquad n = 1, ..., 3. \tag{47}$$

Theorem 4.2 now allows us to make sense of the decomposition (35). Since $\mu_n/(1+\mu_n) \to 0$ as $t \to -\infty$ and $\mu_n/(1+\mu_n) \to 1$ as $t \to \infty$, it follows that the ghost component of u_n defined by (38) controls the formation of ghost particles and that the soliton component of u_n defined by (36) controls the exchange of soliton particle identities.

Theorem 4.8. *($N = 3$) The decay of each soliton component u_n^s is described as follows:*

(i) $u_1^s : s_1^- \to s_3^+$,

(ii) $u_2^s : s_2^- \to s_2^+$,

(iii) $u_3^s : s_3^- \to s_1^+$.

Concluding Remarks. We have shown that for KdV three-solitons a duality exists between soliton particle interaction (Theorem 3.1) and ghost particle interaction (Theorem 4.2) and that both interactions satisfy the same conservation laws. Naturally, these results should extend to an arbitrary number of solitons. In particular, our techniques for analyzing the decay of the largest and smallest eigenvalues of the soliton matrix $A(x,t)$ should easily carry over. However, our analysis of the middle eigenvalue required us to consider separately the two possible orderings of soliton collisions, which must also be done for N-solitons.

Lastly, we mention that our theory of soliton decay also applies to other well known partial differential equations that exhibit soliton behavior, e.g. nonlinear Schrodinger (NLS) and Kadomtsev-Petviashvili (KP) equations. Our preliminary investigations indeed show that such a phenomenon occurs. We shall take up this matter in an upcoming paper.

Appendix A. Moving Frames

It will be useful to write our soliton matrix A given by (4) in several different forms. One form involves writing A in terms of the natural moving frames $\{\nu_n\}$:

$$A = \left(\tfrac{c_mc_n}{k_m+k_n}e^{k_m\nu_m+k_n\nu_n}\right), \qquad m,n = 1,..,N. \tag{48}$$

Then using the relation

$$\nu_n = \nu_l + 4k_n(k_l^2 - k_n^2)t, \tag{49}$$

another form can be gotten for A by writing it purely in terms of fixed moving frame ν_l:

$$A = \left(\tfrac{c_mc_n}{k_m+k_n}e^{(k_m+k_n)\nu_l+4[k_m(k_l^2-k_m^2)+k_n(k_l^2-k_n^2)]t}\right), \qquad m,n = 1,...,N. \tag{50}$$

Next, it will also be useful to write A in terms of an arbitrary ghost moving frame ν_{pq} with $p < q$:

$$A = \left(\tfrac{c_m c_n}{k_m + k_n} e^{(k_m+k_n)\nu_{pq} + 4[k_m(k_{pq}^2 - k_m^2) + k_n(k_{pq}^2 - k_n^2)]t} \right) \tag{51}$$

$$= e^{8(k_p^2 k_q + k_p k_q^2)t} A_{pq}. \tag{52}$$

Here, the matrix A_{pq} appearing in (52) is defined as

$$A_{pq} = \left(\tfrac{c_m c_n}{k_m + k_n} e^{(k_m+k_n)\nu_{pq} + [F_{pq}(k_m) + F_{pq}(k_n)]t} \right) \tag{53}$$

where

$$F_{pq}(x) \equiv x(k_{pq}^2 - x^2) - (k_p^2 k_q + k_p k_q^2). \tag{54}$$

Lemma A.1. *Assume $p < q$. Then*

(i) $F_{pq}(k_p) = F_{pq}(k_q) = 0$.

(ii) $F_{pq}(k_n) < 0$ for $k_n < k_p$ or $k_q < k_n$.

(iii) $F_{pq}(k_n) > 0$ for $k_p < k_n < k_q$.

Proof. Since (i) is trivial, we shall only prove (ii) and (iii). To prove (ii), we first let the reader check that

$$F'_{pq}(x) > 0 \quad \text{for } 0 < x < \sqrt{k_{pq}^2/3} \tag{55}$$

$$F'_{pq}(x) < 0 \quad \text{for } x > \sqrt{k_{pq}^2/3}. \tag{56}$$

Then assuming $k_n < k_p < \sqrt{k_{pq}^2/3}$, it follows from (55) and part (i) of this lemma that $F_{pq}(k_n) < 0$. A similar argument can be used for the case of $\sqrt{k_{pq}^2/3} < k_q < k_n$ to establish $F_{pq}(k_n) < 0$. Part (iii) can be handled in a similar fashion. □

Let $\{\mu_1^{pq}, ..., \mu_N^{pq}\}$ denote the eigenvalues of A_{pq}. Then $\mu_n = e^{8(k_p^2 k_q + k_p k_q^2)t}\mu_n^{pq}$ and so

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\log(1+\mu_n) &= \frac{\partial^2}{\partial x^2}\log[e^{8(k_p^2 k_q + k_p k_q^2)t}(e^{-8(k_p^2 k_q + k_p k_q^2)t} + \mu_n^{pq})] \\ &= \frac{\partial^2}{\partial x^2}\log(e^{-8(k_p^2 k_q + k_p k_q^2)t} + \mu_n^{pq}). \end{aligned} \tag{57}$$

Appendix B. Cauchy Matrices

We recall formulas involving determinants and inverses of soliton matrices which have the form of Cauchy matrices. Let A and A_{pq} given by (4) and (53). Denote by C_{mn} to be the cofactor of a_{mn} in A and define

$$a^{mn} = \prod_{\substack{i<j \\ i=m \text{ or } j=m}}^{N} \frac{1}{(k_i - k_j)} \prod_{\substack{i<j \\ i=n \text{ or } j=n}}^{N} \frac{1}{(k_i - k_j)} \prod_{\substack{i,j \\ i=m \text{ or } j=n}}^{N} (k_i + k_j). \tag{58}$$

Lemma B.1. *(cf.* [PS], *p. 92)*

(i) $\det(A) = \prod_{i<j}^{N}(k_i - k_j)\prod_{i<j}^{N}(k_i - k_j)\prod_{i,j}^{N}\frac{1}{(k_i+k_j)}\prod_{i=1}^{N} c_i^2 e^{2k_i\nu_i}$.

(ii) $C_{mn} = (-1)^{m+n} \prod_{\substack{i<j \\ i,j\neq m}}^{N}(k_i - k_j) \prod_{\substack{i<j \\ i,j\neq n}}^{N}(k_i - k_j) \prod_{\substack{i,j \\ i\neq m, j\neq n}}^{N}\frac{1}{(k_i+k_j)} \prod_{\substack{i=1 \\ i\neq m}}^{N} c_i e^{k_i\nu_i} \prod_{\substack{j=1 \\ j\neq n}}^{N} c_j e^{k_j\nu_j}$.

(iii) $A^{-1} = \left(\frac{a^{mn}}{c_m c_n} e^{-(k_m \nu_m + k_n \nu_n)} \right)$.

(iv) $A^{-1}_{pq} = \left(\frac{a^{mn}}{c_m c_n} e^{-(k_m + k_n)\nu_{pq} - [F_{pq}(k_m) + F_{pq}(k_n)]t} \right)$.

Appendix C. Eigenvalue Asymptotics

We next prove certain asymptotic properties regarding eigenvalues of soliton matrices for $N = 3$. Let A be a soliton matrix with eigenvalues $\{\mu_n\}$ and eigenvectors $\{\mathbf{u}_n\}$ which form an orthonormal basis since A is symmetric.

Lemma C.1. *(i)* $\mu_n \to 0$ *as* $x \to -\infty$ *for* $n = 1, 2, 3$.

(ii) $e^{-2k_n x}\mu_n \to e^{-8k_n^3 t + 2k_n \Delta_n^+}$ *as* $x \to \infty$ *for* $n = 1, 2, 3$.

Proof. Part (i) is obvious since A converges to the zero matrix as $t \to \infty$. We prove (ii) by treating each eigenvalue separately. If μ_1 is an eigenvalue of A, then $e^{-2k_1 x}\mu_1$ is an eigenvalue of $e^{-2k_1 x}A$ which converges to the matrix $e^{-8k_1^3 t + 2k_1 \Delta_1^+} E_{11}$ as $x \to \infty$. It follows that $e^{2k_1 x}\mu_1 \to e^{-8k_1^3 t + 2k_1 \Delta_1^+}$. By a similar argument, the matrix $e^{2k_3 x}A^{-1}$ has $e^{2k_3 x}/\mu_3$ as an eigenvalue and by Lemma B.1-(iii) converges to $e^{-8k_3^3 t + 2k_3 \Delta_3^+} E_{33}$ as $x \to \infty$. This proves that $e^{-2k_3 x}\mu_3$ converges to $e^{-8k_3^3 t + 2k_3 \Delta_3^+}$. Lastly, we use the product formula $\mu_1\mu_2\mu_3 = \det A$, Lemma B.1-(i), and results for μ_1 and μ_3 to easily establish that $e^{-2k_2 x}\mu_2 \to e^{-8k_2^3 t + 2k_2 \Delta_2^+}$. This completes the proof of (ii). □

Next, define $\{e_n\}$ to be the standard unit vectors of length N, $\{E_{mn}\}$ the elementary $N \times N$ matrices, and introduce the vector

$$\mathbf{c} = (c_1(t)e^{k_1 x}, \dots, c_N(t)e^{k_N x})^T. \tag{59}$$

Lemma C.2. *(i)* $\mu_n' = (\mathbf{c}^T \mathbf{u}_n)^2$ *where* $\mu_n' = \partial \mu_n / \partial x$ *for* $n = 1, 2, 3$.

(ii) $\mu_n' \to 0$ *as* $x \to -\infty$ *for* $n = 1, 2, 3$.

(iii) $e^{-2k_n x}\mu_n' \to 2k_n e^{-8k_n^3 t + 2k_n \Delta_n^+}$ *as* $x \to \infty$ *for* $n = 1, 2, 3$.

Proof. To prove (i) we begin with the diagonalization $D = U^{-1}AU$. It follows from the formulas $A' = \mathbf{c}\mathbf{c}^T$ and $U^{-1} = U^T$ that

$$D' = -U^{-1}U'U^{-1}AU + U^{-1}A'U + U^{-1}AU' \tag{60}$$

$$= -U^T U' D + U^T \mathbf{c}\mathbf{c}^T U + DU^T U'. \tag{61}$$

As D is diagonal the matrix $-U^T U' D + DU^T U'$ has zero entries on its main diagonal. This implies that the diagonal entries of D' and $U^T \mathbf{c}\mathbf{c}^T U$ must agree, i.e. $\mu_n' = (\mathbf{c}^T \mathbf{u}_n)^2$, and proves (i). Part (ii) follows immediately from (i) since $\mathbf{c}$ converges to the zero vector as $x \to -\infty$. To prove (iii) we will treat each eigenvalue separately. For μ_1, we have that $\mathbf{u}_1 \to \mathbf{e}_1$ as $x \to \infty$ since $e{-}2k_1 A \to E_{11}$. It follows from (i) that $e^{-2k_1 x}\mu_1' = e^{-2k_1 x}(\mathbf{c}^T \mathbf{u}_1)^2 \to 2k_1 e^{-8k_1^3 t + 2k_n \Delta_1}$. To handle μ_3, we apply (i) to A^{-1} to obtain

$$\left(\frac{1}{\mu_3}\right)' = (\tilde{\mathbf{c}}^T \mathbf{u}_3)^2 \tag{62}$$

where $\tilde{\mathbf{c}}$ is prescribed by Lemma B.1, (iii):

$$\tilde{\mathbf{c}} = (\frac{\tilde{c}_1}{c_1} c_1^{-1}(t)e^{-k_1 x}, \dots, \frac{\tilde{c}_N}{c_N} c_N^{-1}(t)e^{-k_N x})^T. \tag{63}$$

It follows from (62) that $e^{-2k_3 x}\mu_3' = e^{2k_3 x}\mu_3^2(\tilde{\mathbf{c}}^T \mathbf{u}_3)^2 \to 2k_3 e^{-8k_3^2 t + 2k_3 \Delta_3^+}$ as $x \to \infty$. Lastly, we differentiate the product formula $\mu_1\mu_2\mu_3 = \det(A)$ and use results for μ_1' and μ_3' to establish that $e^{-2k_2 x}\mu_2 \to e^{-8k_2^3 t + 2k_2 \Delta_2^+}$. This proves (iii). □

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