A Combinatorial Formula for Certain Two-Dimensional Sequences Related to Generalized Bernoulli Polynomials

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Recurrences for Two-Dimensional Sequences

Triangular Recurrence (Pascal's Triangle):

$$x(p,n) = x(p-1,n) + x(p-1,n-1) \qquad (n \ge p \ge 0)$$

$$x(0,n) = f(n)$$

$$\begin{array}{cccc}
f(0) & f(1) & f(2) & f(3) \\
x(0,0) & x(0,1) & x(0,2) & x(0,3) \\
 & \boxed{x(1,1)} & \boxed{x(1,2)} & x(1,3) \\
 & & \boxed{x(2,2)} & x(2,3) \\
 & & & x(3,3)
\end{array}$$

$$x(0,n) = \boxed{f(n)}$$

$$x(1,n) = x(0,n) + x(0,n-1) = \boxed{f(n) + f(n-1)}$$

$$x(2,n) = x(1,n) + x(1,n-1)$$

$$= [x(0,n) + x(0,n-1)] + [x(0,n-1) + x(0,n-2)]$$

$$= \boxed{f(n) + 2f(n-1) + f(n-2)}$$

$$x(3,n) = \boxed{f(n) + 3f(n-1) + 3f(n-2) + f(n-3)}$$

$$x(p,n) = \binom{p}{0} f(n) + \binom{p}{1} f(n-1) + \ldots + \binom{p}{p} f(n-p) = \boxed{\sum_{k=0}^{p} \binom{p}{k} f(n-k)}$$

Generalization 1:

$$x(p,n) = ax(p-1,n) + bx(p-1,n-1) \qquad (n \ge p \ge 0)$$
$$x(0,n) = f(n)$$

$$x(0,n) = \boxed{f(n)}$$

$$x(1,n) = \boxed{af(n) + bf(n-1)}$$

$$x(2,n) = \boxed{a^2 f(n) + 2abf(n-1) + b^2 f(n-2)}$$

$$x(3,n) = \boxed{a^3 f(n) + 3a^2bf(n-1) + 3ab^2 f(n-2) + b^3 f(n-3)}$$

$$x(p,n) = \binom{p}{0} a^p f(n) + \binom{p}{1} a^{p-1}bf(n-1) + \ldots + \binom{p}{p} b^p f(n-p)$$

$$= \boxed{\sum_{k=0}^{p} \binom{p}{k} a^{n-k} b^k f(n-k)}$$

Generalization 2:

$$x(p,n) = a(p)x(p-1,n) + b(p)x(p-1,n-1) \qquad (n \ge p \ge 0)$$

$$x(0,n) = f(n)$$

$$x(0,n) = \boxed{f(n)}$$

$$x(1,n) = \boxed{a(1)f(n) + b(1)f(n-1)}$$

$$x(2,n) = \boxed{a(2)a(1)f(n) + [a(2)b(1) + b(2)a(1)+]f(n-1) + b(2)b(1)f(n-2)}$$

$$x(3,n) = \boxed{\begin{aligned} &a(3)a(2)a(1)f(n) \\ &+[a(3)a(2)b(1) + a(3)b(2)a(1) + b(3)a(2)a(1)]f(n-1) \\ &+[a(3)b(2)b(1) + b(3)a(2)b(1) + b(3)b(2)a(1)]f(n-2) \\ &+b(3)b(2)b(1)f(n-3) \end{aligned}}$$

$$x(p,n) = ?$$

-How many *a*’s and *b*’s in each term?
- How to assign indices to *a*’s and *b*’s?

Subsets of $\{1,2,\ldots,p\}$

$$
\begin{array}{llll}
\underline{x(3,n)}: & \underline{\text{Position of } b\text{'s}}: & \underline{\text{Subset of } \{1,2,3\}} & \\
f(n): & a(1)a(2)a(3) & -\varnothing & \text{(0-element subset)} \\
f(n-1): & b(1)a(2)a(3) & \left.\begin{array}{l} -\{1\} \\ -\{2\} \\ -\{3\} \end{array}\right\} & \text{(1-element subsets)} \\
 & a(1)b(2)a(2) & & \\
 & a(1)a(2)b(3) & & \\
f(n-2): & b(1)b(2)a(3) & \left.\begin{array}{l} -\{1,2\} \\ -\{1,3\} \\ -\{2,3\} \end{array}\right\} & \text{(2-element subsets)} \\
 & b(1)a(2)b(3) & & \\
 & a(1)b(2)b(3) & & \\
f(n-3): & b(1)b(2)b(3) & -\{1,2,3\} & \text{(3-element subset)}
\end{array}
$$

Closed Formula for $x(p,n)$

$A_p(k) = \{\sigma \subseteq \{1,2,...,p\} : |\sigma| = k\}$ (e.g. $\{1,2,3\}$)

$\sigma = \{i_1, i_2, ..., i_k\} \in A_p(k)$ (e.g. $\sigma = \{1,2\}$)

$\bar{\sigma} =$ complement of σ in $\{1,2,...,k\} = \{j_1, j_2, ..., j_{p-k}\}$ (e.g. $\bar{\sigma} = \overline{\{1,2\}} = \{3\}$)

$b(\sigma) = b(i_1)b(i_2)\cdots b(i_k)$ (e.g. $b(\{1,2\}) = b(1)b(2)$)

$a(\bar{\sigma}) = a(j_1)a(j_2)\cdots a(j_{p-k})$ (e.g. $a(\overline{\{1,2\}}) = a(3)$)

Terms of $x(p,n)$ are of the form $a(\bar{\sigma})b(\sigma)f(n-k)$

(e.g. $a(\overline{\{1,2\}})b(\{1,2\})f(n-2) = b(1)b(2)a(3)f(n-2)$)

$$\boxed{x(p,n) = \sum_{k=0}^{p}\left(\sum_{\sigma \in A_p(k)} a(\bar{\sigma})b(\sigma)\right) f(n-k)}$$

Generalization 3:

$$x(p,n) = a(n)x(p-1,n) + b(n)x(p-1,n-1) \qquad (n \ge p \ge 0)$$
$$x(0,n) = f(n)$$

$$x(0,n) = \boxed{f(n)}$$

$$x(1,n) = \boxed{a(n)f(n) + b(n)f(n-1)}$$

$$x(2,n) = \boxed{\begin{array}{l} a(n)a(n)f(n) + [a(n)b(n) + b(n)a(n-1)]f(n-1) \\ +b(n)b(n-1)f(n-2) \end{array}}$$

$$x(3,n) = \boxed{\begin{array}{l} a(n)a(n)a(n)f(n) \\ +[a(n)a(n)b(n) + a(n)b(n)a(n-1) + b(n)a(n-1)a(n-1)]f(n-1) \\ +[a(n)b(n)b(n-1) + b(n)a(n-1)b(n-1) + b(n)b(n-1)a(n-2)]f(n-2) \\ +b(n)b(n-1)b(n-2)f(n-3) \end{array}}$$

-How many *a*'s and *b*'s in each term?
- How to shift indices for *a*'s and *b*'s?

Subsets of $\{1,2,\ldots,p\}$

$\underline{x(3,n)}$:	$\underline{\text{Position of } b\text{'s}}$:	$\underline{\text{Subset of } \{1,2,3\}}$	
$f(n)$:	$a(n)a(n)a(n)$	$-\ \varnothing$	(0-element subset)
$f(n-1)$:	$b(n)a(n)a(n)$	$-\ \{1\}$	
	$a(n-1)b(n)a(n)$	$-\ \{2\}$	(1-element subsets)
	$a(n-1)a(n-1)b(n)$	$-\ \{3\}$	
$f(n-2)$:	$b(n-1)b(n)a(n)$	$-\ \{1,2\}$	
	$b(n-1)a(n-1)b(n)$	$-\ \{1,3\}$	(2-element subsets)
	$a(n-2)b(n-1)b(n)$	$-\ \{2,3\}$	
$f(n-3)$:	$b(n-2)b(n-1)b(n)$	$-\ \{1,2,3\}$	(3-element subset)

Shift by Position Index

$\underline{x(n,3)}$: $\underline{\text{Position index (rank) with respect to } \sigma}$

$$f(n): \quad a(n)a(n)a(n) \quad -\{1,2,3\}$$

$$f(n-1): \quad b(n)a(n)a(n) \quad -\{1,2,3\}$$

$$\overset{1<2}{a(n-1)}\,b(n)a(n) \quad -\{1,2,3\}$$

$$\overset{1<3}{a(n-1)}\,\overset{2<3}{a(n-1)}\,b(n) \quad -\{1,2,3\}$$

$$f(n-2): \quad \overset{1<2}{b(n-1)}\,b(n)a(n) \quad -\{1,2,3\}$$

$$\overset{1<3}{b(n-1)}\,\overset{2<3}{a(n-1)}\,b(n) \quad -\{1,2,3\}$$

$$\overset{1<2,1<3}{a(n-2)}\,\overset{2<3}{b(n-1)}\,b(n) \quad -\{1,2,3\}$$

$$f(n-3): \quad \overset{1<2,1<3}{b(n-2)}\,\overset{2<3}{b(n-1)}\,b(n) \quad -\{1,2,3\}$$

Closed Formula for $x(p,n)$

$\sigma = \{i_1, i_2, ..., i_k\} \in A_p(k)$ (e.g. $\sigma = \{1,2\} \subset \{1,2,3\}$)

$\bar{\sigma} =$ complement of σ in $\{1,2,...,k\} = \{j_1, j_2, ..., j_{p-k}\}$ (e.g. $\bar{\sigma} = \overline{\{1,2\}} = \{3\}$)

Position index function (rank of an element with respect to a set):

$I(j;\sigma) = \left|\{i \in \sigma : j < i\}\right|$ (e.g. $I(1;\{1,2\}) = 1, I(2;\{1,2\}) = 0, I(3;\{1,2\}) = 0$)

$b_I(\sigma) = b(n - I(i_1,\sigma))b(n - I(i_2,\sigma)) \cdots b(n - I(i_k,\sigma))$

(e.g. $b_I(\{1,2\}) = b(n - I(1,\{1,2\}))b(n - I(2,\{1,2\}))] = b(n-1)b(n)$)

$a_I(\bar{\sigma}) = a(n - I(j_1,\sigma))a(n - I(j_2,\sigma)) \cdots a(n - I(j_{p-k},\sigma))$

(e.g. $a_I(\overline{\{1,2\}}) = a_I(\{3\}) = a(n - I(3,\{1,2\})) = a(n)$)

Terms of $x(p,n)$ are of the form $a_I(\bar{\sigma})b_I(\sigma)f(n-k)$

(e.g. $a_I(\overline{\{1,2\}})b_I(\{1,2\})f(n-2) = b(n-1)b(n)a(n)f(n-2)$)

$$\boxed{x(p,n) = \sum_{k=0}^{p}\left(\sum_{\sigma \in A_p(k)} a_I(\bar{\sigma})b_I(\sigma)\right)f(n-k)}$$

Final Generalization:

$$x(p,n) = a(p,n)\,x(p-1,n) + b(p,n)\,x(p-1,n-1) \qquad (n \geq p \geq 0)$$

$$x(0,n) = f(n)$$

$$x(0,n) = \boxed{f(n)}$$

$$x(1,n) = \boxed{a(1,n)\,f(n) + b(1,n)\,f(n-1)}$$

$$x(2,n) = \boxed{\begin{array}{l} a(1,n)a(2,n)\,f(n) \\ +[b(1,n)\,a(2,n) + a(1,n-1)\,b(2,n)]\,f(n-1) \\ +b(1,n-1)b(2,n)\,f(n-2) \end{array}}$$

$$x(3,n) = \boxed{\begin{array}{l} a(1,n)a(2,n)\,a(3,n)\,f(n) \\ +[b(1,n)a(2,n)a(3,n) + a(1,n-1)b(2,n)a(3,n) \\ \qquad\qquad + a(1,n-1)a(2,n-1)b(3,n)]\,f(n-1) \\ +[b(1,n-1)b(2,n)a(3,n) + b(1,n-1)a(2,n-1)b(3,n) \\ \qquad\qquad + a(1,n-2)b(2,n-1)b(3,n)]\,f(n-2) \\ +b(1,n-2)b(2,n-1)b(3,n)\,f(n-3) \end{array}}$$

$$b_I(\sigma) = b(i_1, n - I(i_1,\sigma))b(i_2, n - I(i_2,\sigma))\cdots b(i_k, n - I(i_k,\sigma))$$

(e.g. $b_I(\{1,2\}) = b(1,n-1)b(2,n)$)

$$a_I(\bar{\sigma}) = a(j_1, n - I(j_1,\sigma))a(j_2, n - I(j_2,\sigma))\cdots a(j_{p-k}, n - I(j_{p-k},\sigma))$$

(e.g. $a_I(\overline{\{1,2\}}) = a_I(\{3\}) = a(3,n)$)

Terms of $x(p,n)$ are of the form $a_I(\bar{\sigma})b_I(\sigma)f(n-k)$

(e.g. $a_I(\overline{\{1,2\}})b_I(\{1,2\})f(n-2) = b(1,n-1)b(2,n)a(3,n)f(n-2)$)

Theorem: Suppose

$$x(p,n) = a(p,n)x(p-1,n) + b(p,n)x(p-1,n-1) \qquad (n \geq p \geq 0)$$

$$x(0,n) = f(n)$$

Then

$$\boxed{x(p,n) = \sum_{k=0}^{p}\left(\sum_{\sigma\in A_p(k)} a_I(\bar{\sigma})b_I(\sigma)\right) f(n-k)}$$

Proof: Induction on p

Application to Bernoulli Polynomials

Classical Bernoulli Polynomials:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6}$$

Generalized Bernoulli Polynomials (GBPs) of Order *p*:

$$\frac{t^p e^{xt}}{(e^t - 1)^p} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!}$$

NOTE : $B_n^{(1)}(x) = B_n(x)$ (classical Bernoulli polynomials)

Lemma: (Nörlund, 1924) For $n \geq 1, p > 1$

$$\underbrace{B_n^{(p)}(x)}_{x(p,n)} = \underbrace{\left(1 - \frac{n}{p-1}\right)}_{a(p,n)} B_n^{(p-1)}(x) + n \underbrace{\left(\frac{x}{p-1} - 1\right)}_{b(p,n)} B_{n-1}^{(p-1)}(x)$$

Closed Formula for GBPs

Theorem: (Dilcher, 1996) For $n \geq p \geq 1$

$$B_n^{(p)}(x) = (-1)^{p-1} p \binom{n}{p} \sum_{k=0}^{p-1} \frac{(-1)^k}{n-k} \left(\sum_{m=0}^{k} \binom{p-1-k+m}{m} \underbrace{s(p, p-k+m)}_{\text{Stirling numbers of the first kind}} x^m \right) B_{n-k}(x)$$

Combinatorial analog of Dilcher's formula based on our results for generalized triangular recurrences:

Theorem: For $n \geq p \geq 1$

$$B_n^{(p)}(x) = \sum_{k=0}^{p-1} \left(\sum_{\sigma \in A_{p-1}(k)} a_I(\bar{\sigma}) b_I(\sigma) \right) B_{n-k}(x)$$

Further Generalizations

1. Hypergeometric Bernoulli Polynomials
2. Three-term recurrences

References

[1] K. Dilcher, *Sums of Products of Bernoulli Numbers*, J. Number Theory **60** (1996), 23-41.

[2] A. Hassen and H. D. Nguyen, *Hypergeometric Bernoulli Polynomials and Appell Sequences* , Intern. J. Number Theory **4** (2008), No. 5, pp. 767-774.

[3] N. E. Nörlund, *Vorlesungen über Differenzenrechnung.* Berlin, Germany: Springer-Verlag, 1924.