# Appell Sequences and Hypergeometric Bernoulli Polynomials

Abdul Hassen and Hieu D. Nguyen*
Rowan University
Glassboro, NJ

Joint Math Meetings
New Orleans, LA
January 5, 2007

# Bernoulli Polynomials

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

...

Generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Recursive formula

$$\sum_{k=0}^{n} \frac{B_k(x)}{k!(n+1-k)!} = \frac{x^n}{n!}$$

The Bernoulli *numbers* are defined by $B_n = B_n(0)$

$$B_0 = 1, \quad B_1 - \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \; ...$$

# Some Properties of Bernoulli Numbers and Polynomials

1. $$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n} \quad \left( \zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \right)$$

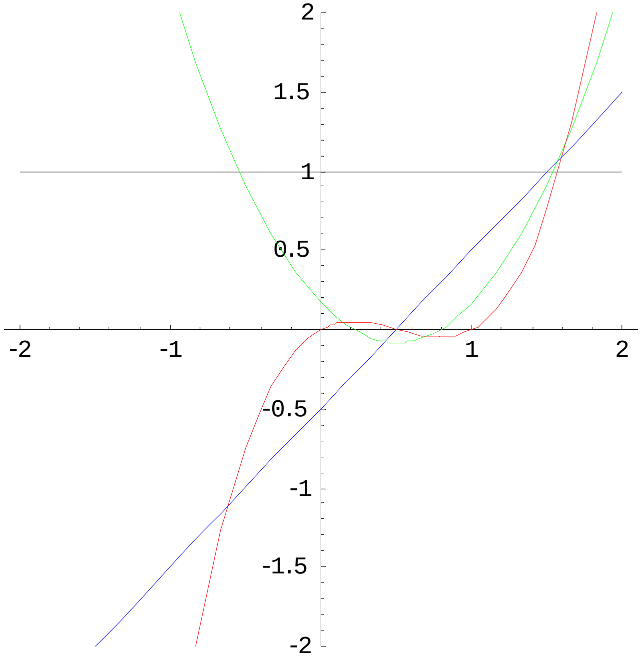
2. $$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$$

3. $$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

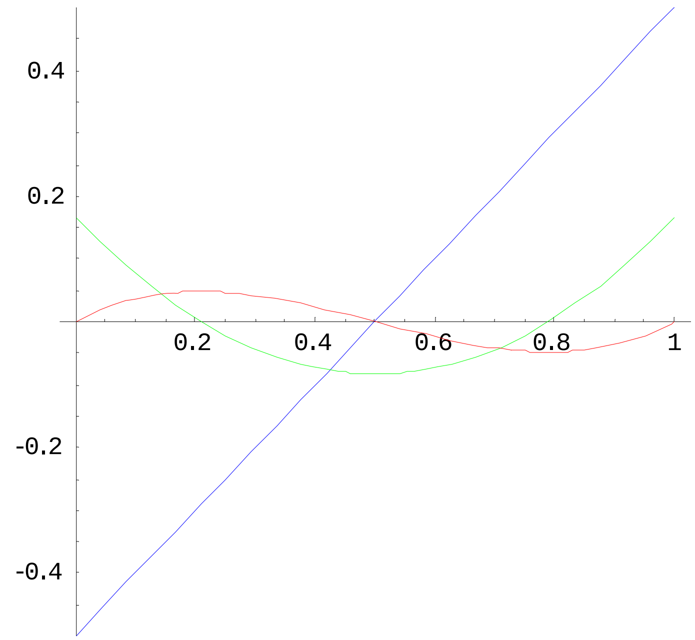
4. $$B_n(1-x) = (-1)^n B_n(x)$$

5. $$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n$$

# What about derivative and integral properties?




Interval [-2,2]




Interval [0,1]

$B_0(x)$ $B_1(x)$ $B_2(x)$ $B_3(x)$

**Theorem**: (Appell, 1882) The following two definitions of Bernoulli polynomials are equivalent:

I. Generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

II. Appell sequence with zero mean:

(i) $B_0(x) = 1$

(ii) $B_{n+1}'(x) = (n+1) B_n(x)$

(iii) $\int_0^1 B_n(x)\,dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

# Hypergeometric Bernoulli Polynomials ($N$ = 2)

F. Howard (1977)

$$\frac{t^2 e^{xt}/2!}{e^t - 1 - t} = \sum_{n=0}^{\infty} B_n(2,x)\frac{t^n}{n!}$$

Recursive formula

$$\sum_{k=0}^{n} \frac{B_k(2,x)}{k!(3)_{n-k}} = \frac{x^n}{n!}, \qquad (3)_m = \begin{cases} 0 & m = 0 \\ 3\cdot 4\cdots(3+m-1) & m > 0 \end{cases}$$

$$B_0(2,x) = 1, \qquad B_1(2,x) = x - \frac{1}{3},$$

$$B_2(2,x) = x^2 - \frac{2}{3}x + \frac{1}{18}, \quad B_3(2,x) = x^3 - x^2 + \frac{1}{6}x + \frac{1}{90}.$$

# Some Properties of Hypergeometric Bernoulli Numbers and Polynomials

1. $\zeta_2(-n) = (-1)^{n+1} \dfrac{2B_{n+1}(2)}{n(n+1)}$ $\quad \left( \zeta_2(s) = \dfrac{1}{\Gamma(s+1)} \int_0^\infty \dfrac{x^s}{e^x - 1 - x} dx \right)$

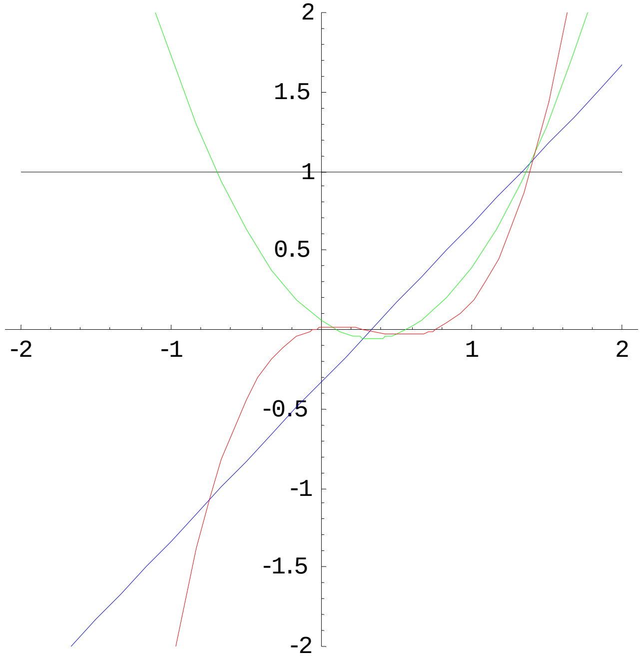
2. $B_n(2,x) = \sum_{k=0}^{n} \binom{n}{k} B_k(2) \cdot x^{n-k}$

3. $\sum_{k=0}^{n-1} \dfrac{B_k(2)}{k!(n-k+2)!} = \begin{cases} 1/2! & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$

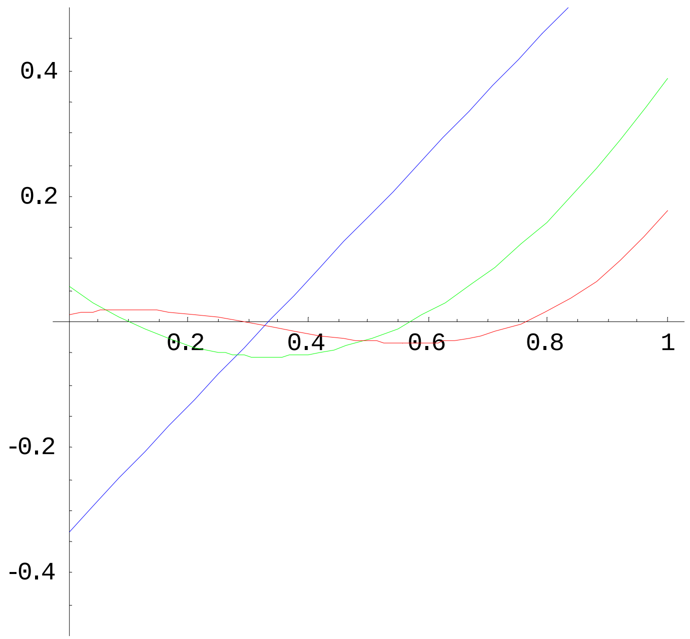
We define *hypergeometric Bernoulli numbers* by

$$B_n(2) = B_n(2,0)$$

# Again, what about derivative and integral properties?




Interval [-2,2]

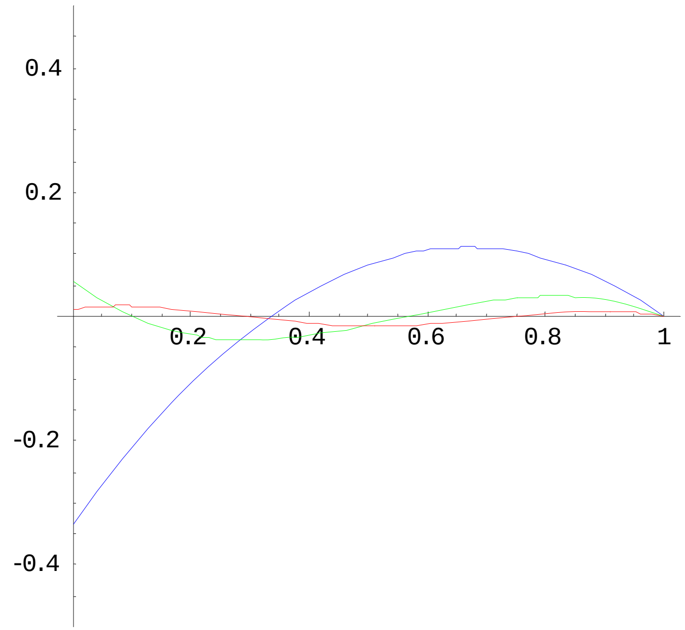



Interval [0,1]

$B_0(x)$ $B_1(x)$ $B_2(x)$ $B_3(x)$

Weighted means:

$$(1-x)B_n(2,x)$$




Interval [0,1]

$B_0(x)$ $B_1(x)$ $B_2(x)$ $B_3(x)$

**Theorem**: ($N$ = 2) The following two definitions of hypergeometric Bernoulli polynomials $B_n(2,x)$ are equivalent:

I. Generating function:

$$\frac{t^2 e^{xt}/2!}{e^t - 1 - t} = \sum_{n=0}^{\infty} B_n(2,x)\frac{t^n}{n!}$$

II. Appell sequence with zero first moment:

(i) $B_0(2,x) = 1$

(ii) $B_{n+1}'(2,x) = (n+1)B_n(2,x)$

(iii) $\int_0^1 (1-x)B_n(2,x)dx = \begin{cases} 1/2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

## Proof of Theorem $(\mathrm{I} \Rightarrow \mathrm{II})$

(ii)

$$\frac{d}{dx}\left[\frac{t^2 e^{xt}/2!}{e^t-1-t}\right]=\frac{d}{dx}\left[\sum_{n=0}^{\infty}B_n(2,x)\frac{t^n}{n!}\right]$$

$$\Rightarrow t\left(\frac{t^2 e^{xt}/2!}{e^t-1-t}\right)=\sum_{n=1}^{\infty}B_n^{'}(2,x)\frac{t^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty}B_n(2,x)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\frac{B_{n+1}^{'}(2,x)}{(n+1)}\cdot\frac{t^{n+1}}{n!}$$

$$\therefore B_n(2,x)=\frac{B_{n+1}{}'(2,x)}{n+1}$$

(iii)

$$\int_0^1 (1-x)B_n(2,x)dx = \int_0^1 (1-x)\left[\sum_{k=0}^{n}\binom{n}{k}B_k(2)\cdot x^{n-k}\right]dx$$

$$= \sum_{k=0}^{n}\binom{n}{k}B_k(2)\left(\int_0^1 (1-x)x^{n-k}dx\right)$$

$$= \sum_{k=0}^{n}\frac{n!B_k(2)}{k!(n-k)!}\cdot\frac{1}{(n-k+1)(n-k+2)}$$

$$= n!\sum_{k=0}^{n}\frac{B_k(2)}{k!(n-k+2)!}$$

$$= \begin{cases} 1/2 & \text{if } n=0 \\ 0 & \text{if } n>0 \end{cases}$$

$(\mathrm{II} \Rightarrow \mathrm{I})$ Define

$$G(x,t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Then

$$\frac{\partial G}{\partial x} = \sum_{n=1}^{\infty} B_n^{'}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} n B_{n-1}(x) \frac{t^n}{n!} = t \sum_{n=0}^{\infty} B_{n-1}(x) \frac{t^n}{n!} = tG(x,t)$$

This implies

$$G(x,t) = e^{tx} g(t)$$

It follows that

$$\int_0^1 G(x,t)dx = \int_0^1 \left[ \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \right] dx$$

$$\Rightarrow \int_0^1 e^{xt} g(t) dx = \sum_{n=0}^{\infty} \left( \int_0^1 B_n(x) dx \right) \frac{t^n}{n!}$$

$$\Rightarrow g(t) \frac{e^t - 1}{t} = 1 \qquad \Rightarrow g(t) = \frac{t}{e^t - 1} \qquad \therefore G(x,t) = \frac{te^{xt}}{e^t - 1}$$

# Hypergeometric Bernoulli Polynomials

**Theorem**: The following two definitions of hypergeometric Bernoulli polynomials $B_n(N,x)$ are equivalent for *all positive integers N*:

I. Generating function:

$$\frac{t^N e^{xt}/N!}{e^t-1-t}=\sum_{n=0}^{\infty}B_n(N,x)\frac{t^n}{n!}$$

II. Appell sequence with zero first moment:

(i) $B_0(N,x)=1$

(ii) $B_{n+1}'(N,x)=(n+1)B_n(N,x)$

(iii) $\int_0^1(1-x)B_n(N,x)dx=\begin{cases}1/N & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}$

# Bernoulli-Padé Polynomials

K. Dilcher and L. Malloch (2002)

$$\frac{(-1)^s r!s!t^{r+s+1}e^{xt}}{(r+s)!(r+s+1)!\left[Q^{(r,s)}(t)e^t - P^{(r,s)}(t)\right]} = \sum_{n=0}^{\infty} B_n^{(r,s)}(x)\frac{t^n}{n!}$$

Padé Approximants for $e^{\mathrm{t}}$

$$\frac{P^{(r,s)}(t)}{Q^{(r,s)}(t)} \approx e^t$$

$$P^{(r,s)}(t) = \sum_{j=0}^{r} \frac{(r+s-j)!}{(r+s)!} t^j, \qquad Q^{(r,s)}(t) = \sum_{j=0}^{s} \frac{(r+s-j)!}{(r+s)!}(-t)^j$$

Bernoulli-Padé versus Hypergeometric Bernoulli

$$B_n^{(r,s)}(x) = B_n(N,x), \qquad (r = N-1,\ s=0)$$

**Theorem**: The following two definitions of Bernoulli-Padé polynomials $B_n^{(r,s)}(x)$ are equivalent for *all non-negative integers r and s*:

I. Generating function:

$$\frac{(-1)^s r!s!t^{r+s+1}e^{xt}}{(r+s)!(r+s+1)!\left[Q^{(r,s)}(t)e^t - P^{(r,s)}(t)\right]} = \sum_{n=0}^{\infty} B_n^{(r,s)}(x)\frac{t^n}{n!}$$

II. Appell sequence with zero first moment:

(i) $B_0^{(r,s)}(x) = 1$ (ii) $B_{n+1}^{(r,s)}(x) = (n+1)B_n^{(r,s)}(x)$

(iii) $\int_0^1 x^s (1-x)^r B_n^{(r,s)}(x)dx = \begin{cases} r!s!/(r+s+1)! & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

# References


[1] Appell, P. E. "Sur une classe de polynomes." *Annales d'École Normal Superieur, Ser. 2* **9**, 119-144, 1882.

[2] K. Dilcher and L. Malloch, Arithmetic Properties of Bernoulli-Pade Numbers and Polynomials, J. Number Theory 92 (2002), 330-347.

[3] A. Hassen and H. D. Nguyen, Hypergeometric Zeta Functions, Preprint, 2005. Available at the Mathematics ArXiv: http://arxiv.org/abs/math.NT/0509637

[4] F. T. Howard, Numbers Generated by the Reciprocal of e^x-1-x, Math. Comput. 31 (1977) No. 138, 581-598.