How Bernoulli Did It:

Sums of Powers, Generating Functions, and Bernoulli Numbers

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Rowan Math Seminar
October 11, 2006

MAA Column: Ed Sandifer's *How Euler Did it:*

http://www.maa.org/news/howeulerdidit.html

Bernoulli Challenge

Prize Problem: Find the sum of the 10^{th} powers of the first 1000 natural numbers, i.e.

$$1^{10} + 2^{10} + 3^{10} + ... + 1000^{10}$$

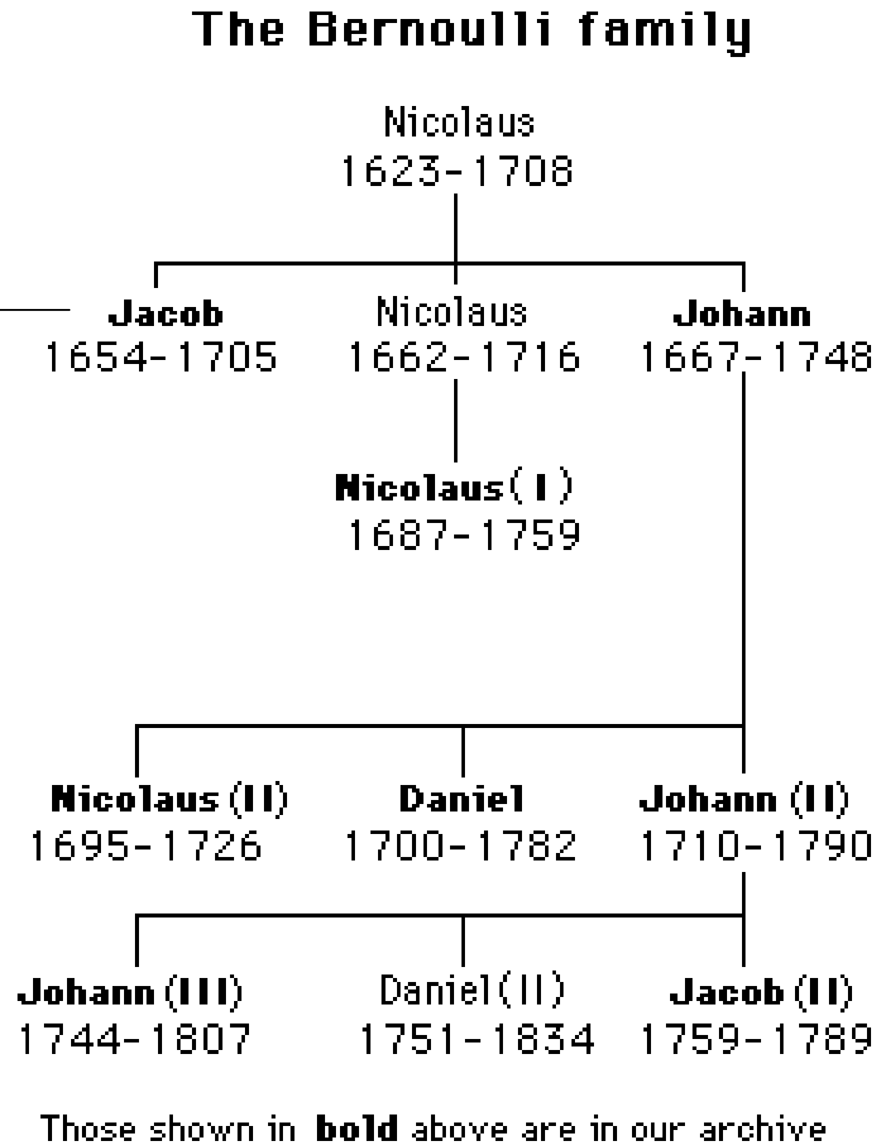
Rules: Students only. No technology allowed.

Hint: The answer is a 32-digit number.

(Bernoulli did it "in less than half of a quarter of an hour")

Jacob (Jacques) Bernoulli

1654 - 1705



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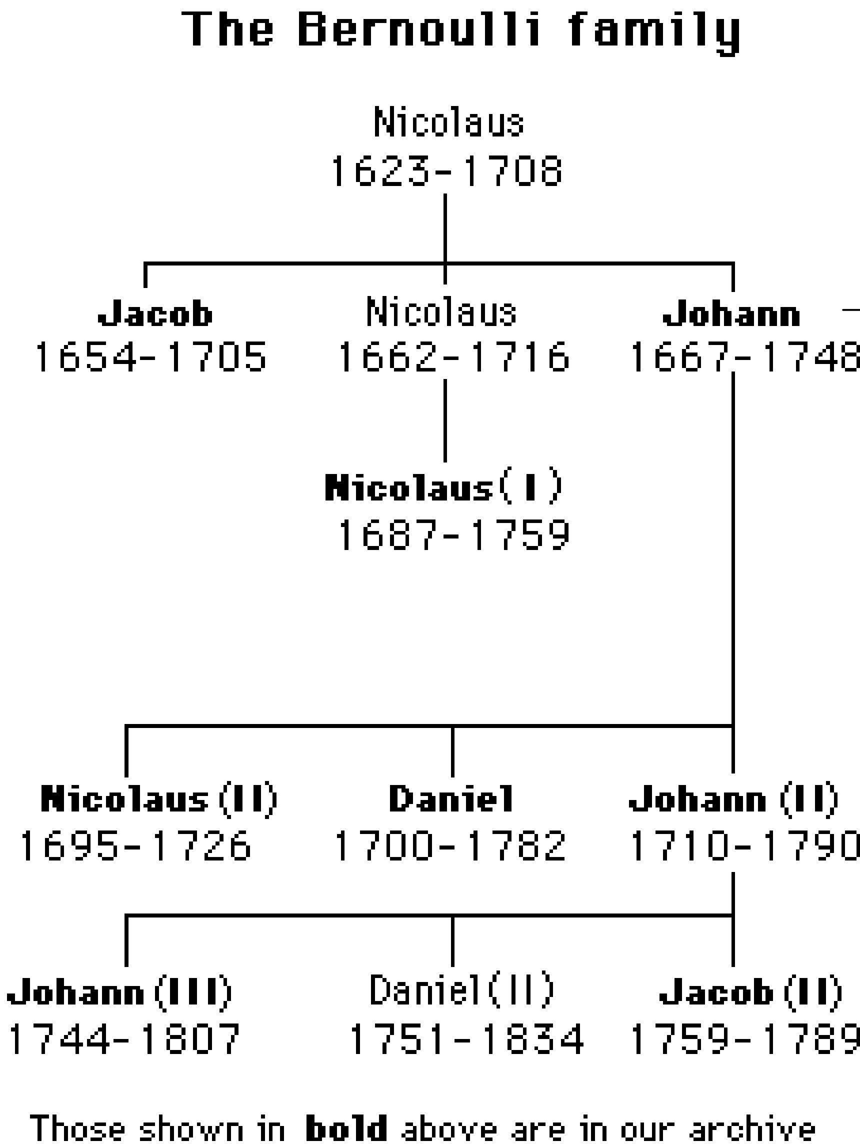

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Leonhard Euler

1707 - 1783

Sums of Powers

$$1+2+3+\ldots+98+99+100=50(101)=5050$$

$$1^2+2^2+3^2+\ldots+98^2+99^2+100^2=?$$

Johann Faulhaber, *Academia Algebrae* (1631)

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2} = N \qquad \text{(Pythagoreans)}$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = M \qquad \text{(Archimedes)}$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2 = N^2 \qquad \text{(Nichomacus)}$$

$$\sum_{k=1}^{n} k^4 = 1^4+2^4+\ldots+n^4 = M\left(-\frac{1}{5}+\frac{6}{5}N\right) \qquad \text{(Haytham)}$$

…

$$\sum_{k=1}^{n} k^{17} = 1^{17} + 2^{17} + \ldots + n^{17} = (1280N^9 - 6720N^8 + 21220N^7$$
$$- 46880N^6 + 72912N^5 - 74220N^4$$
$$+ 43404N^3 - 10851N^2)/45$$

Faulhaber's Formulas

$$\sum_{k=1}^{n} k^{2r} = M(b_1 + b_2 N + \ldots + b_r N^{r-1}) \qquad \text{(even powers)}$$

$$\sum_{k=1}^{n} k^{2r+1} = N^2(c_1 + c_2 N + \ldots + c_r N^{r-1}) \qquad \text{(odd powers greater than 1)}$$

Jacob Bernoulli, *Ars Conjectandi* (1713)

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n^2}{2}+\frac{n}{2} \qquad \frac{1}{2}+\frac{1}{2}=1$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6} \qquad \frac{1}{3}+\frac{1}{2}+\frac{1}{6}=1$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+\ldots+n^3 = \frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4} \qquad \frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1$$

$$\sum_{k=1}^{n} k^4 = 1^4+2^4+\ldots+n^4 = \frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30} \qquad \frac{1}{5}+\frac{1}{2}+\frac{1}{3}-\frac{1}{30}=1$$

$$\sum_{k=1}^{n} k^5 = 1^5+2^5+\ldots+n^5 = \frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12} \qquad \frac{1}{6}+\frac{1}{2}+\frac{5}{12}-\frac{1}{12}=1$$

$$\sum_{k=1}^{n} k^p = 1^p+2^p+\ldots+n^p = ?$$

$$p=1: \quad \frac{1}{2}, \quad \frac{1}{2}$$

$$p=2: \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{1}{6}=\frac{2}{2}\cdot\frac{1}{6}$$

$$p=3: \quad \frac{1}{4}, \quad \frac{1}{2}, \quad \frac{1}{4}=\frac{3}{2}\cdot\frac{1}{6} \quad , \quad 0$$

$$p=4: \quad \frac{1}{5}, \quad \frac{1}{2}, \quad \frac{1}{3}=\frac{4}{2}\cdot\frac{1}{6} \quad , \quad 0, \quad \frac{1}{30}=\xcancel{\frac{4}{2}\cdot\frac{1}{30}}=\frac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}\cdot\frac{1}{30}$$

$$p=5: \quad \frac{1}{6}, \quad \frac{1}{2}, \quad \frac{5}{12}=\frac{5}{2}\cdot\frac{1}{6} \quad , \quad 0, \quad \frac{1}{12}=\frac{5}{2}\cdot\frac{1}{30}=\frac{5\cdot 4\cdot 3}{2\cdot 3\cdot 4}\cdot\frac{1}{30} \quad , \quad 0$$

$$\frac{1}{p+1}, \quad \frac{1}{2}, \quad ,\frac{p}{2}A, \quad \frac{p(p-1)(p-2)}{2\cdot 3\cdot 4}B, \quad \frac{p(p-1)(p-2)(p-3)(p-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6}C,\ldots$$

$$A=\frac{1}{6}, B=-\frac{1}{30}, C=\frac{1}{42},\ldots$$

Bernoulli's formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{p}{2} A n^{p-1} + \frac{p(p-1)(p-2)}{2\cdot 3\cdot 4} B n^{p-3}$$
$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6} C n^{p-3} + \ldots$$

(Original) Bernoulli numbers

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, \ldots$$

Modern formula

$$\sum_{k=1}^{n} k^p = \frac{p!}{0!(p+1)!} B_0 n^{p+1} + \frac{p!}{1!p!} B_1 n^p + \frac{p!}{2!(p-1)!} B_2 n^{p-1} + \frac{p!}{3!(p-2)!} B_3 n^{p-2} + \ldots$$
$$= \sum_{k=0}^{p} (-1)^{\delta_{kp}} \frac{p!}{k!(p+1-k)!} B_k n^{p+1-k}$$

$$B_0 = 1, \boxed{B_1 = \frac{1}{2}}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \ldots$$

Figurate Numbers

"*It appears to me that if one wants to make progress in mathematics one should study the masters, and not the pupils*.“ --- Niels Abel (1802-1829)

Passage from *Ars Conjectandi* (p. 95)

We will observe in passing that, many [scholars] engaged in the contemplation of figurate numbers (among them Faulhaber and Remmelin of Ulm, Wallis, Mercator, in his Logarithmotechnia *and others) but I do not know of one who gave a general and scientific proof of this property*.

Wallis in his Arithmetica Infinitorum *investigated by means of induction the ratios that the series of squares, cubes, and other powers of natural numbers have to the series of terms each equal to the greatest term. This he put in the foundation of his method.*

His next step was to establish 176 properties of trigonal, pyramidal, and other figurate numbers, but it would have been better and more fitting to the nature of the subject if the process would have been reversed and he would have first given a discussion of figurate numbers, demonstrated in a general and accurate way, and only then have proceeded with the investigation of the sums of powers. Even disregarding the fact that the method of induction is not sufficiently scientific and, moreover, requires special work for every new series; it is a method of common judgment that the simpler and more primitive things should precede others. Such are the figurate numbers as related to the powers, since they are formed by addition, while the others are formed by multiplication;…

Translated from the Latin by Professor Jekuthiel Ginsburg, Yeshiva College, New York City.

Bernoulli’s Triangle (Pascal’s) of Figurate Numbers (Binomial Coefficients)

$(n=1)$	1	0	0	0	0	0	0	0
	1	1	0	0	0	0	0	0
	1	2	1	0	0	0	0	0
	1	3	3	1	0	0	0	0
	1	4	6	4	1	0	0	0
	1	5	10	10	5	1	0	0
	...	...	...	...	...	...	...	0
	A_n	B_n	C_n	D_n	E_n	...		

$$A_n = 1,$$

$$B_n = n-1,$$

$$C_n = \frac{(n-1)(n-2)}{1\cdot 2},$$

$$D_n = \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3},$$

...

Figurate Identities

$$A_1 + A_2 + \ldots + A_{n-1} = B_n$$

$$B_1 + B_2 + \ldots + B_{n-1} = C_n$$

$$C_1 + C_2 + \ldots + C_{n-1} = D_n$$

...

Bernoulli's proof of his formula based on figurate identities:

<u>Case p = 1:</u>

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n^2}{2}+\frac{n}{2}$$

Assume the figurate identity $B_1 + B_2 + \ldots + B_{n-1} = C_n$ holds:

$$0+1+2+\ldots+(n-1) = \frac{n(n-1)}{1\cdot 2} = \frac{n^2}{2}-\frac{n}{2}$$

Then

$$\sum_{k=1}^{n} k = \sum_{k=1}^{n}(k-1+1) = \sum_{k=1}^{n}(k-1)+\sum_{k=1}^{n}1$$

$$= \left(\frac{n^2}{2}-\frac{n}{2}\right)+n = \frac{n^2}{2}+\frac{n}{2}$$

Note: Observe that Bernoulli's proof resembles the modern one using the principle of mathematical induction even though he himself viewed induction as not "sufficiently scientific".

Case p = 2:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

Assume the figurate identity $C_1 + C_2 + \ldots + C_{n-1} = D_n$ holds:

$$0 + 0 + 1 + 3 + 6 + \ldots + \frac{(n-1)(n-2)}{1 \cdot 2} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \frac{n^3 - 3n^2 + 2n}{6}$$

Then since

$$\sum_{k=1}^{n} \frac{(k-1)(k-2)}{1 \cdot 2} = \sum_{k=1}^{n} \frac{1}{2} k^2 - \sum_{k=1}^{n} \frac{3}{2} k + \sum_{k=1}^{n} 1$$

and

$$\sum_{k=1}^{n} \frac{3k}{2} = \frac{3}{4} n^2 + \frac{3}{4} n,$$

$$\sum_{k=1}^{n} 1 = n$$

it follows that

$$\sum_{k=1}^{n}\frac{1}{2}k^2=\left(\frac{n^3-3n^2+2n}{6}\right)+\left(\frac{3}{4}n^2+\frac{3}{4}n\right)-n$$

$$=\frac{n^3}{6}+\frac{n^2}{4}+\frac{n}{12}$$

Hence,

$$\sum_{k=1}^{n}k^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

<u>Case p = 3:</u>

$$\sum_{k=1}^{n}k^3=1^3+2^3+\ldots+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

Assume the figurate identity $D_1+D_2+\ldots+D_{n-1}=E_n$ holds:

$$0+0+0+1+4+10+\ldots+\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}=\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

$$=\frac{n^4-6n^3+11n^2-6n}{6}$$

Then since

$$\sum_{k=1}^{n}\frac{(k-1)(k-2)(k-3)}{1\cdot 2\cdot 3}=\sum_{k=1}^{n}\frac{1}{6}k^3-\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}\frac{11}{6}k-\sum_{k=1}^{n}1$$

and

$$\sum_{k=1}^{n}k^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6},$$

$$\sum_{k=1}^{n}\frac{11}{6}k=\frac{11}{12}n^2+\frac{11}{12}n,$$

$$\sum_{k=1}^{n}1=n$$

it follows that

$$\sum_{k=1}^{n}\frac{1}{6}k^3=\left(\frac{n^4-6n^3+11n^2-6n}{6}\right)+\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)-\left(\frac{11}{12}n^2+\frac{11}{12}n\right)+n$$

$$=\frac{n^4}{24}+\frac{n^3}{12}+\frac{n^2}{24}$$

Hence,

$$\sum_{k=1}^{n}k^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}$$

Proof for arbitrary p?

Unfortunately Bernoulli does not provide one, but the recipe for an inductive proof is evident in his demonstration for $p = 1, 2, 3$.

http://www.jstor.org/view/0025570x/di021151/02p0123y/0

Bernoulli's formula for $p = 10$:

$$\sum_{k=1}^{n} k^{10} = \frac{1}{66}\left(6n^{11} + 33n^{10} + 55n^{9} - 66n^{7} + 66n^{5} - 33n^{3} + 5n\right)$$

Generating Functions

Leonard Euler (1755)

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \frac{1}{30240}t^6 + \ldots$$

$$= 1 + \left(-\frac{1}{2}\right)\frac{t}{1!} + \left(\frac{1}{6}\right)\frac{t^2}{2!} + \left(-\frac{1}{30}\right)\frac{t^4}{4!} + \left(\frac{1}{42}\right)\frac{t^6}{6!} + \ldots$$

$$= B_0 + B_1\frac{t}{1!} + B_2\frac{t^2}{2!} + B_4\frac{t^4}{4!} + B_6\frac{t^6}{6!} + \ldots$$

Bernoulli numbers

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

$$B_0 = 1, \boxed{B_1 = -\frac{1}{2}}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, \ldots$$

Equate series coefficients of *t*:

$$t = (e^t - 1)\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \left(\sum_{m=1}^{\infty} \frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}\right)$$

$$= \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} B_k \frac{t^{m+k}}{m!k!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B_k}{k!(n+1-k)!}\right) t^{n+1} \qquad (n+1 = m+k)$$

Recursive formula:

$$\sum_{k=0}^{n} \frac{B_k}{k!(n+1-k)!} = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$n = 0: \boxed{B_0 = 1}$$

$$n = 1: \frac{B_0}{2} + B_1 = 0 \Rightarrow \boxed{B_1 = -\frac{1}{2}}$$

$$n = 2: \frac{B_0}{6} + \frac{B_1}{2} + \frac{B_2}{2} = 0 \Rightarrow \boxed{B_2 = \frac{1}{6}}$$

Bernoulli Polynomials

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Equate series coefficients of *t*:

$$te^{xt} = (e^t - 1) \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

$$t \sum_{m=0}^{\infty} \frac{(xt)^n}{n!} = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} B_k(x) \frac{t^{m+k}}{m!k!}$$

$$\sum_{n=0}^{\infty} \frac{x^n t^{n+1}}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k(x)}{k!(n+1-k)!} \right) t^{n+1} \qquad (n+1 = m+k)$$

Recursive formula

$$\sum_{k=0}^{n} \frac{B_k(x)}{k!(n+1-k)!} = \frac{x^n}{n!}$$

$$n = 0: \boxed{B_0(x) = 1}$$

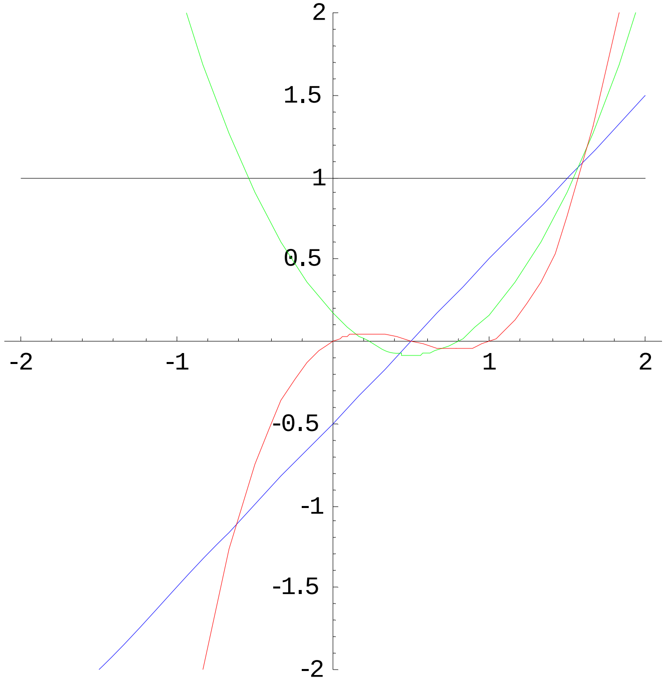
$$n = 1: \frac{B_0(x)}{2} + B_1(x) = x \Rightarrow \boxed{B_1(x) = x - \frac{1}{2}}$$

$$n = 2: \frac{B_0(x)}{6} + \frac{B_1(x)}{2} + \frac{B_2(x)}{2} = \frac{x^2}{2} \Rightarrow \boxed{B_2(x) = x^2 - x + \frac{1}{6}}$$

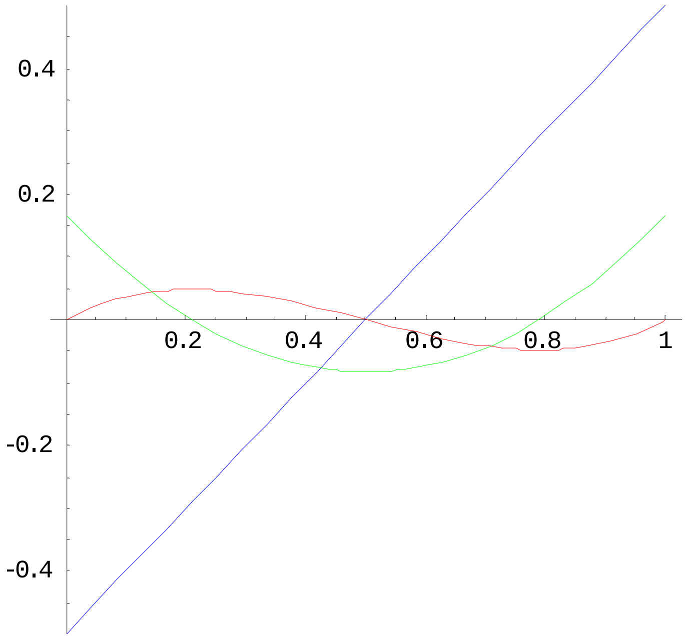
$$n = 3: \boxed{B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x}$$

Observe that

$$B_n = B_n(0)$$



Interval [-2,2]



Interval [0,1]

$B_0(x)$

$B_1(x)$

$B_2(x)$

$B_3(x)$

Some Properties of Bernoulli Numbers and Polynomials

1. $$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}$$

2. $$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

3. $$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n$$

4. $$B_n(1-x) = (-1)^n B_n(x)$$

5. $$\sum_{k=0}^{n-1} k^{p-1} = \frac{1}{p}\left[B_p(n) - B_p(0) \right]$$

6. $$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$$

What about derivative and integral properties?

Appell Sequences

Derivative property

$$B_1'(x) = \frac{d}{dx}(x - 1/2) = 1 = B_0(x)$$

$$B_2'(x) = \frac{d}{dx}(x^2 - x + \frac{1}{6}) = 2x - 1 = 2\left(x - \frac{1}{2}\right) = 2B_1(x)$$

$$B_3'(x) = \frac{d}{dx}(x^3 - \frac{3}{2}x^2 + \frac{1}{2}x) = 3x^2 - 3x + \frac{1}{2} = 3\left(x^2 - x + \frac{1}{6}\right) = 3B_2(x)$$

Integral property:

$$\int_0^1 B_1(x)dx = \int_0^1 \left(x - \frac{1}{2}\right)dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_0^1 = 0$$

$$\int_0^1 B_2(x)dx = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)dx = 0$$

$$\int_0^1 B_2(x)dx = \int_0^1 \left(x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\right)dx = 0$$

Theorem: (Appell?) The following two definitions of Bernoulli polynomials are equivalent:

I. Generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

II. Appell sequence with zero mean:

(i) $B_0(x) = 1$

(ii) $B_{n+1}'(x) = (n+1) B_n(x)$

(iii) $\int_0^1 B_n(x)dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

Explicit construction of Bernoulli polynomials as an Appell sequence:

$$B_1'(x) = B_0(x) \Rightarrow B_1(x) = \int B_0(x)dx = \int 1dx = x + C_1$$

$$\int_0^1 B_1(x)dx = 0 \Rightarrow \int_0^1 (x + C_1)dx = 0 \Rightarrow \left[\frac{1}{2}x^2 + C_1 x\right]_0^1 = 0$$

$$\Rightarrow \frac{1}{2} + C_1 = 0 \Rightarrow C_1 = -\frac{1}{2} = B_1$$

$$\therefore B_1(x) = B_0 x + B_1$$

$$\therefore B_2(x) = B_0 x^2 + 2B_1 x + B_2$$

$$\therefore B_3(x) = B_0 x^3 + 3B_1 x^2 + 3B_2 x + B_4$$

General formula

$$B_n(x) = B_0 x^n + \frac{n}{1} B_1 x^{n-1} + \frac{n(n-1)}{1 \cdot 2} B_2 x^{n-2} + \ldots + B_n$$

Proof of Theorem $(\Rightarrow)$

$$\frac{d}{dx}\left[\frac{te^{xt}}{e^t-1}\right]=\frac{d}{dx}\left[\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\right]\Rightarrow t\left(\frac{te^{xt}}{e^t-1}\right)=\sum_{n=1}^{\infty}B_n^{'}(x)\frac{t^n}{n!}$$

$$\Rightarrow\sum_{n=0}^{\infty}B_n(x)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}B_{n+1}^{'}(x)\frac{t^{n+1}}{(n+1)!}$$

$$\therefore\frac{B_{n+1}{}'(x)}{n+1}=B_n(x)$$

$$\int_0^1 B_n(x)dx=\int_0^1\left[\sum_{k=0}^{n}\binom{n}{k}B_k\cdot x^{n-k}\right]dx=\sum_{k=0}^{n}\binom{n}{k}B_k\left(\int_0^1 x^{n-k}dx\right)$$

$$=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{B_k}{n-k+1}=n!\sum_{k=0}^{n}\frac{B_k}{k!(n-k+1)!}$$

$$=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}\quad\text{(Recursive formula)}$$

Hypergeometric Bernoulli Polynomials

$$\frac{t^2 e^{xt}/2}{e^t - 1 - t} = \sum_{n=0}^{\infty} B_n(2,x) \frac{t^n}{n!}$$

Recursive formula

$$\sum_{k=0}^{n} \frac{B_k(2,x)}{k!(3)_{n-k}} = \frac{x^n}{n!}, \qquad (3)_m = \begin{cases} 0 & m = 0 \\ 3 \cdot 4 \cdots (3+m-1) & m > 0 \end{cases}$$

$$n = 0: \boxed{B_0(2,x) = 1}$$

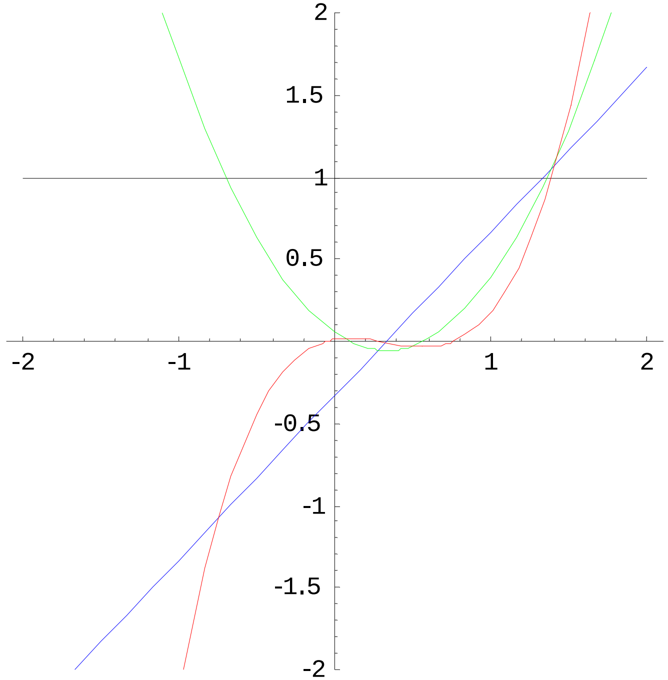
$$n = 1: \boxed{B_1(2,x) = x - \frac{1}{3}}$$

$$n = 2: \boxed{B_2(2,x) = x^2 - \frac{2}{3}x + \frac{1}{18}}$$

$$n = 3: \boxed{B_3(2,x) = x^3 - x^2 + \frac{1}{6}x + \frac{1}{90}}$$

We can define *hypergeometric Bernoulli numbers* as

$$B_n(2) = B_n(2,0)$$



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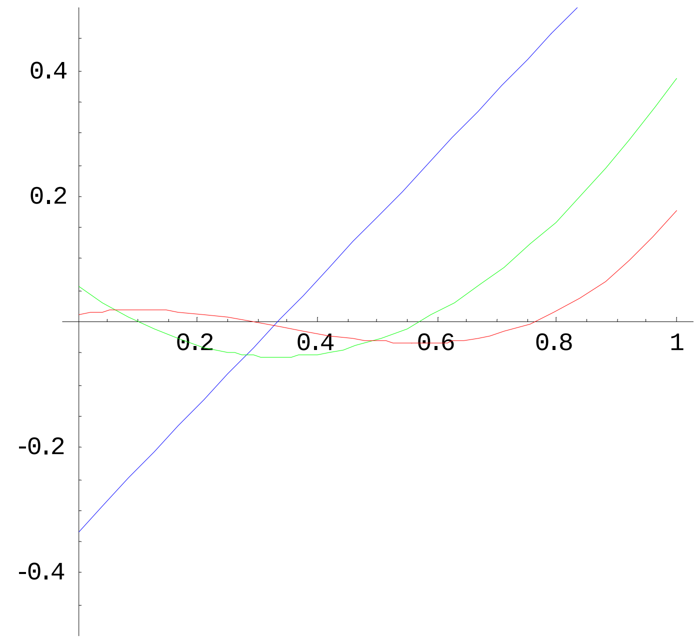


Interval [0,1]

$B_0(x)$

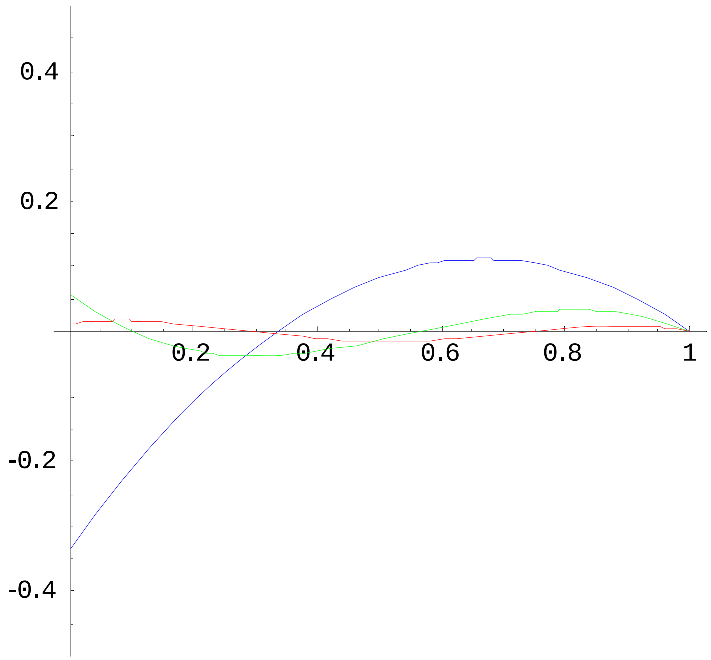
$B_1(x)$

$B_2(x)$

$B_3(x)$

Weighted means:

$$(1-x)B_n(2,x)$$



$$\int_0^1 (1-x)B_1(2,x)dx = \int_0^1 (1-x)(x-1/3)dx = \int_0^1 (-x^2+4x/3-1/3)dx$$

$$= \left[-x^3/3+2x^2/3-x/3\right]_0^1 = 0$$

Theorem: The following two definitions of hypergeometric Bernoulli polynomials are equivalent:

I. Generating function:

$$\frac{t^2 e^{xt}/2}{e^t - 1 - t} = \sum_{n=0}^{\infty} B_n(2,x)\frac{t^n}{n!}$$

II. Appell sequence with zero first moment:

(i) $B_0(x) = 1$

(ii) $B_{n+1}'(x) = (n+1)B_n(x)$

(iii) $\int_0^1 (1-x)B_n(x)dx = \begin{cases} 1/2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

Answer to Prize Problem:

$$\sum_{k=1}^{n} k^{10} = \frac{1}{66}\left(6n^{11} + 33n^{10} + 55n^{9} - 66n^{7} + 66n^{5} - 33n^{3} + 5n\right)$$

$$\sum_{k=1}^{1000} k^{10} = \frac{1}{66}(6\cdot 1000^{11} + 33\cdot 1000^{10} + 55\cdot 1000^{9}$$

$$- 66\cdot 1000^{7} + 66\cdot 1000^{5} - 33\cdot 1000^{3} + 5\cdot 1000)$$

$$= \frac{1}{66}(6033055000000000066000000000005000$$

$$- 6600000000003300000000)$$

$$= \frac{1}{66}(6033054999934000065999967000005000)$$

$$= 91{,}409{,}924{,}241{,}424{,}243{,}424{,}241{,}924{,}242{,}500$$

"With the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum

91409924241424243424241924242500

From this it will become clear how useless was the work of Ismael Bullialdus spent on the compilation of his voluminous Arithmetica Infinitorum *in which he did nothing more than compute with immense labor the sums of the first six powers, which is only a part of what we have accomplished in the space of a single page."*

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