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Jacob Bernoulli
1654-1705

Bernoulli Number Identities via Euler-Maclaurin Summation

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Math Dept Colloquium
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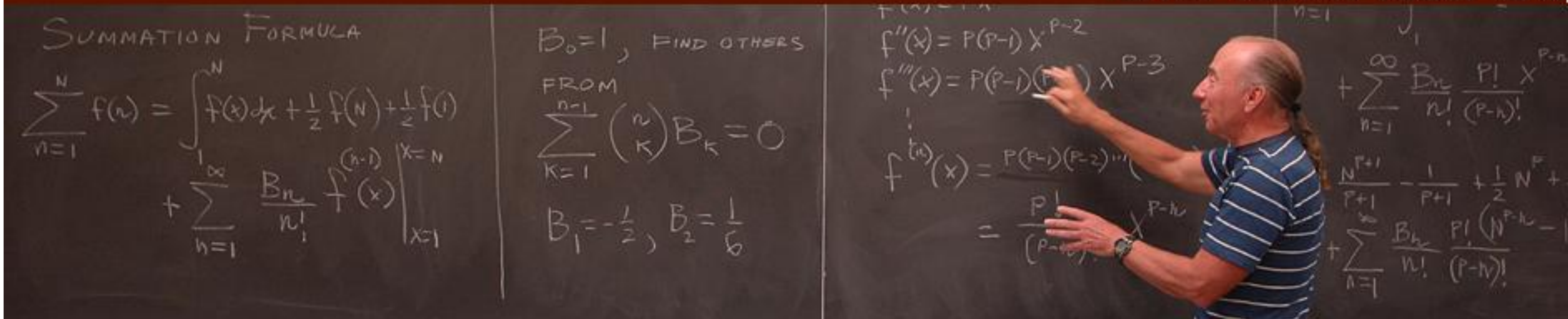

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Sums of Powers

$$1+2+3+\ldots+1000 = \frac{1000(1001)}{2} = 500{,}500$$

$$1^2+2^2+3^2+\ldots+100^2 = \frac{1000(1001)(2001)}{6} = 333{,}833{,}500$$

$$1^{10}+2^{10}+3^{10}+\ldots+1000^{10} = ?$$

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2} = \frac{1}{2}(n^2+n)$$ (Pythagoreans)

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3+3n^2+n)$$ (Archimedes)

$$\sum_{k=1}^{n} k^{10} = \frac{1}{66}\; 6n^{11}+33n^{10}+55n^9-66n^7+66n^5-33n^3+5n$$

Bernoulli Challenge

(October 2006)

Prize Problem: Find the sum of the 10^{th} powers of the first 1000 natural numbers, i.e.

$$1^{10} + 2^{10} + 3^{10} + ... + 1000^{10}$$

Rules: Students only. No technology allowed.

Hint: The answer is a 32-digit number.

(Jacob Bernoulli did it "in less than half of a quarter of an hour")

"With the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum

91409924241424243424241924242500

From this it will become clear how useless was the work of Ismael Bullialdus spent on the compilation of his voluminous Arithmetica Infinitorum *in which he did nothing more than compute with immense labor the sums of the first six powers, which is only a part of what we have accomplished in the space of a single page."*

Jacob Bernoulli, *Ars Conjectandi* (1713)

Sums of Powers Formula

$$\sum_{k=1}^{n} k^p = \frac{p!}{0!(p+1)!} B_0 n^{p+1} - \frac{p!}{1!\,p!} B_1 n^p + \frac{p!}{2!(p-1)!} B_2 n^{p-1}$$

$$+ \frac{p!}{3!(p-2)!} B_3 n^{p-2} + \ldots + B_p n$$

Bernoulli numbers:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \ \ldots$$

$$\sum_{k=1}^{n} k^2 = \frac{2!}{0!3!} B_0 n^{p+1} - \frac{2!}{1!2!} B_1 n^p + \frac{2!}{2!1!} B_2 n^{p-1}$$

$$= \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

$$= \frac{1}{6}(2n^3 + 3n^2 + n)$$

Bernoulli Polynomials

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

...

Generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Recursive formula

$$\sum_{k=0}^{n} \frac{B_k(x)}{k!(n+1-k)!} = \frac{x^n}{n!}$$

The Bernoulli *numbers* are defined by $B_n = B_n(0)$:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \frac{1}{30240}t^6 + ...$$

Some Properties of Bernoulli Numbers and Polynomials

1. $$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n} \qquad \left(\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \right)$$

2. $$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{5}{2} = \frac{5!}{2!3!} = 10$$

3. $$B_n(1-x) = (-1)^n B_n(x) \quad (\Rightarrow B_n(1) = (-1)^n B_n(0))$$

$$B_2(x) = x^2 - x + \frac{1}{6} = x(x-1) + \frac{1}{6}$$

$$B_2(1-x) = (1-x)(-x) + \frac{1}{6} = x(x-1) + \frac{1}{6} = B_2(x)$$

4. $$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n$$

Bernoulli Number Identities

Euler coined the term ‘Bernoulli’ numbers in his textbook E212, *Institutiones calculi differentialis*, Part II, Chapter 5, (1755)

122. Numeri isti per universam serierum doctrinam amplissimum habent usum. Primum enim ex his numeris formari possunt ultimi termini in summis potestatum parium, quos non aeque ac reliquos terminos ex summis praecedentium reperiri posse supra annotavimus. In potestatibus enim paribus postremi summarum termini sunt x per certos numeros multiplicati; qui numeri pro potestatibus II; IV; VI; VIII; &c. sunt $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$, &c. signis alternantibus affecti. Oriuntur autem hi numeri si valores litterarum α, β, γ, δ, &c. supra inventi respective dividantur per numeros impares 3, 5, 7, &c. unde isti numeri, qui ab Inventore *Iacobo Bernoullio* vocari solent Bernoulliani erunt:

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A}$$
$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B}$$
$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C}$$
$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D}$$
$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E}$$

122. These numbers have great use throughout the entire theory of series. First, one can obtain from them the final terms in the sums of even powers, for which we noted above (in §63 of part one) that one cannot obtain them, as one can the other terms, from the sums of earlier powers. For the even powers, the last terms of the sums are products of x and certain numbers, namely for the 2nd, 4th, 6th, 8th, etc., $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$ etc. with alternating signs. But these numbers arise from the values of the letters α, β, γ, δ, etc., which we found earlier, when one divides them by the odd numbers 3, 5, 7, 9, etc. These numbers are called the Bernoulli numbers after their discoverer Jakob Bernoulli,

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A} = B_2$$
$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B} = -B_4$$
$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C} = B_6$$
$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D} = -B_8$$
$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E} = B_{10}$$

123. Thus one immediately obtains the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. from the following equations:

$$\mathfrak{A} = \frac{1}{6}$$

$$\surd \quad \mathfrak{B} = \frac{4\cdot 3}{1\cdot 2} \cdot \frac{1}{5}\mathfrak{A}^2$$

$$\mathfrak{C} = \frac{6\cdot 5}{1\cdot 2} \cdot \frac{2}{7}\mathfrak{A}\mathfrak{B}$$

$$\mathfrak{D} = \frac{8\cdot 7}{1\cdot 2} \cdot \frac{2}{9}\mathfrak{A}\mathfrak{C} + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{1}{9}\mathfrak{B}^2$$

$$\mathfrak{E} = \frac{10\cdot 9}{1\cdot 2} \cdot \frac{2}{11}\mathfrak{A}\mathfrak{D} + \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{2}{11}\mathfrak{B}\mathfrak{C}$$

$$\mathfrak{F} = \frac{12\cdot 11}{1\cdot 2} \cdot \frac{2}{13}\mathfrak{A}\mathfrak{E} + \frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{2}{13}\mathfrak{B}\mathfrak{D} + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} \cdot \frac{1}{13}\mathfrak{C}^2$$

$$\mathfrak{G} = \frac{14\cdot 13}{1\cdot 2} \cdot \frac{2}{15}\mathfrak{A}\mathfrak{F} + \frac{14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{2}{15}\mathfrak{B}\mathfrak{E} + \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} \cdot \frac{2}{15}\mathfrak{C}\mathfrak{D}$$

etc.,

Quadratic recurrence:

$$\sum_{k=0}^{m} \binom{2m}{2k} B_{2k} B_{2(m-k)} = -(2m-1)B_{2m} \qquad (m>1)$$

$$m=2: \quad B_0B_4 + 6B_2B_2 + B_0B_4 = -3B_4 \Rightarrow -5B_4 = 6B_2^2$$

$$\Rightarrow -B_4 = \frac{6}{5}B_2^2 = \frac{4\cdot 3}{1\cdot 2}\cdot\frac{1}{5}B_2^2$$

Euler's proof (trigonometric identity):

$$s = \frac{1}{2}\cot\left(\frac{1}{2}u\right) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}u^{2n-1} = \frac{1}{u} + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}u^{2n-1}$$

$$\frac{ds}{du} = -\frac{1}{4}\csc^2\left(\frac{1}{2}u\right) = -\frac{1}{u^2} + \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)2^{2n} B_{2n}}{(2n)!}u^{2n-2}$$

$$4\frac{ds}{du} + 1 + 4ss = 0 \qquad (\cot^2 u + 1 = \csc^2 u)$$

$$\frac{4}{u^2}\left[-1 + \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)2^{2n} B_{2n}}{(2n)!}u^{2n}\right] + 1 + \frac{4}{u^2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}u^{2n}\right]^2 = 0$$

$$-1 + \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)2^{2n} B_{2n}}{(2n)!}u^{2m} + \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(-1)^m 2^{2m} B_{2k} B_{2(m-k)}}{(2k)!(2(m-k))!}u^{2m} = -\frac{u^2}{4}$$

$$\sum_{m=1}^{\infty}\left[\frac{(-1)^m(2m-1)2^{2m} B_{2m}}{(2m)!} + \sum_{k=0}^{m}\frac{(-1)^m 2^{2m} B_{2k} B_{2(m-k)}}{(2k)!(2(m-k))!}\right]u^{2m} = -\frac{u^2}{4}$$

$$\frac{(-1)^m (2m-1) 2^{2m} B_{2m}}{(2m)!} + \sum_{k=0}^{m} \frac{(-1)^m 2^{2m} B_{2k} B_{2(m-k)}}{(2k)!(2(m-k))!} = 0 \qquad (m>1)$$

$$\sum_{k=0}^{m} \binom{2m}{2k} B_{2k} B_{2(m-k)} = -(2m-1) B_{2m} \qquad (m>1)$$

Alternate Proof (Euler-Maclaurin Summation)

Power Rule:

$$p_n(x) = x^n$$

$$p_n'(x) = n x^{n-1} = \frac{n!}{(n-1)!} p_{n-1}(x)$$

$$p_n''(x) = n(n-1) x^{m-2} = \frac{n!}{(n-2)!} p_{n-2}(x)$$

$$p_n'''(x) = n(n-1)(n-2) x^{m-3} = \frac{n!}{(n-3)!} p_{n-3}(x)$$

$$p_n^{(k)}(x) = \frac{n!}{(n-k)!} x^{n-k} = \frac{n!}{(n-k)!} p_{n-k}(x)$$

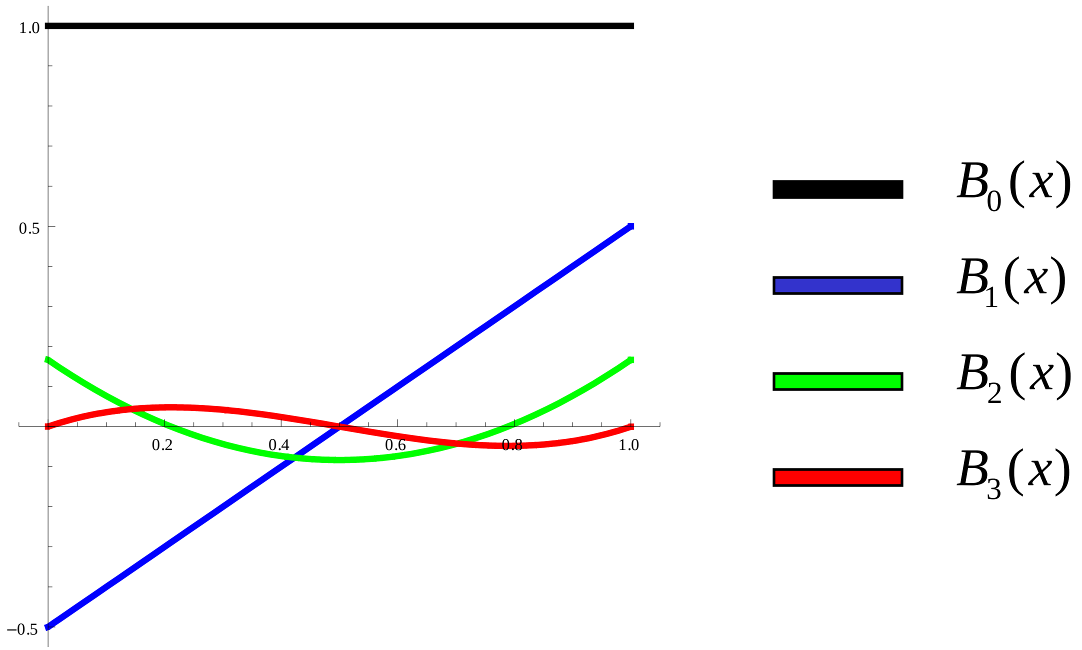
Appell Sequences

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

$$B_1'(x) = 1 = B_0(x)$$

$$B_2'(x) = 2x - 1 = 2(x - 1/2) = 2B_1(x)$$

$$B_3'(x) = 3x^2 - 3x + 1/2 = 3(x^2 - x + 1/6) = 3B_2(x)$$



Bernoulli polynomials as Appell sequence:

(i) $B_0(x) = 1$

(ii) $B_n'(x) = nB_{n-1}(x)$ $\qquad \left(\int B_{n-1}(x)dx = \frac{1}{n} B_n(x) \right)$

(iii) $\int_0^1 B_n(x)dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

Example 1:

$$B_1(x) = \int B_0(x)dx = \int 1dx = x + C = \boxed{x - \frac{1}{2}}$$

$$0 = \int_0^1 B_1(x)dx = \int_0^1 (x+C)dx = \frac{1}{2}x^2 + Cx\Big|_0^1 = \frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$$

Euler-Maclaurin Summation (EMS)

$$\sum_{k=1}^{n-1} f(k) = \int_0^n f(x)dx - \frac{1}{2}[f(n) + f(0)] + \sum_{k=1}^{q} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]$$

$$+ \frac{nB_{2q+2}}{(2q+1)!} f^{(2q+2)}(\alpha), \quad \alpha \in [0, n]$$

Example 2: $f(x) = x$

$$\sum_{k=0}^{n-1} f(k) = \sum_{k=0}^{n-1} k \qquad \int_0^n f(x)dx = \left.\frac{x^2}{2}\right|_0^n = \frac{n^2}{2} \qquad \frac{1}{2}[f(n) + f(0)] = \frac{n}{2}$$

$$\sum_{k=1}^{q} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] = 0 = \frac{nB_{2q+1}}{(2q+1)!} f^{(2q+2)}(\alpha)$$

EMS: $$\sum_{k=0}^{n-1} k = \frac{n^2}{2} - \frac{n}{2} \Rightarrow \sum_{k=0}^{n} k = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

Special Case ($n = 1$)

Repeated Integration by Parts

$$\int_0^1 f(x)dx = \int_0^1 B_0(x) f(x)dx$$

$$u = f(x), \qquad dv = B_0(x)dx$$

$$du = f'(x)dx, \quad v = \int B_0(x)dx = B_1(x)$$

$$= [B_1(x) f(x)]_0^1 - \int_0^1 B_1(x) f'(x)dx$$

$$= [B_1(1) f(1) - B_1(0) f(0)] - \left[\frac{B_2(x) f'(x)}{2!}\right]_0^1 + \frac{1}{2!}\int_0^1 B_2(x) f''(x)dx$$

$$= -B_1[f(1) + f(0)] - \frac{1}{2!}[B_2(1) f'(1) - B_2(0) f'(0)] + \left[\frac{B_3(x) f''(x)}{3!}\right]_0^1$$

$$- \frac{1}{3!}\int_0^1 B_3(x) f'''(x)dx$$

$$= -B_1[f(1) + f(0)] - \frac{B_2}{2!}[f'(1) - f'(0)] + \frac{1}{3!}[B_3(1) f''(1) - B_3(0) f''(0)]$$

$$- \left[\frac{B_4(x) f'''(x)}{4!}\right]_0^1 + \frac{1}{4!}\int_0^1 B_4(x) f^{(4)}(x)dx$$

...

$$= -B_1[f(1)+f(0)] - \frac{B_2}{2!}[f'(1)-f'(0)] - \frac{B_3}{3!}[f''(1)+f''(0)]$$

$$-\frac{1}{4!}[B_4(1)f'''(1) - B_4(0)f'''(0)] + \left[\frac{B_5(x)f^{(4)}(x)}{5!}\right]_0^1 - \frac{1}{5!}\int_0^1 B_5(x)f^{(5)}(x)dx$$

...

Assume *f* a polynomial of degree *p*:

$$\int_0^1 f(x)dx = -\sum_{k=1}^{p+1}\frac{B_k}{k!}[f^{(k-1)}(1) + (-1)^{k-1}f^{(k-1)}(0)]$$

$$\boxed{\int_0^1 f(x)dx = \frac{1}{2}[f(1)+f(0)] - \sum_{k=2}^{p}\frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]}$$

$$\left(\sum_{k=1}^{n-1} f(k) = \int_0^n f(x)dx - \frac{1}{2}[f(n)+f(0)] + \sum_{k=1}^{q}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]\right.$$

$$\left.+\frac{nB_{2q+1}}{(2q+1)!}f^{(2q+2)}(\alpha), \quad \alpha\in[0,n]\right)$$

Example 3: $f(x) = x^{m-1} \quad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 x^{m-1}dx = \frac{1}{m} \qquad \frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[1+0] = \frac{1}{2}$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = \frac{(m-1)!}{(m-k)!}x^{m-k}\Big|_{x=1} - \frac{(m-1)!}{(m-k)!}x^{m-k}\Big|_{x=0}$$

$$= \begin{cases} 0 & \text{if } m-k=0 \text{ (or } k=m) \\ \dfrac{(m-1)!}{(m-k)!} & \text{otherwise} \end{cases}$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1) + f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$\frac{1}{m} = \frac{1}{2} - \sum_{k=2}^{m-1} \frac{B_k}{k!}\left[\frac{(m-1)!}{(m-k)!}\right] \Rightarrow 1 - \frac{m}{2} + \sum_{k=2}^{m-1} \frac{m!}{k!(m-k)!}B_k = 0$$

$$\boxed{\sum_{k=0}^{m-1}\binom{m}{k}B_k = 0}$$

(Recall formula: $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k \cdot x^{n-k}$)

Example 3: $f(x) = B_m(x) \quad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 B_m(x)dx = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m > 0 \end{cases}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[B_m(1) + B_m(0)] = \begin{cases} B_m & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

$$= B_m$$

Derivatives of Bernoulli polynomials

$$B_m'(x) = mB_{m-1}(x) = \frac{m!}{(m-1)!} B_{m-1}(x)$$

$$B_m^{(k)}(x) = \frac{m!}{(m-k)!} B_{m-k}(x)$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = B_m^{(k-1)}(1) - B_m^{(k-1)}(0)$$

$$= \frac{m!}{(m-(k-1))!}[B_{m-(k-1)}(1) - B_{m-(k-1)}(0)]$$

$$= \begin{cases} -2\frac{m!}{(m-k+1)!}B_{m-k+1} & \text{if } m+k-1 \text{ odd} \\ 0 & \text{if } m+k-1 \text{ even} \end{cases}$$

$$= \begin{cases} -2m!B_1 & \text{if } m-k+1=1 \text{ (or } k=m) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} m! & \text{if } k=m \\ 0 & \text{otherwise} \end{cases}$$

For $m > 1$:

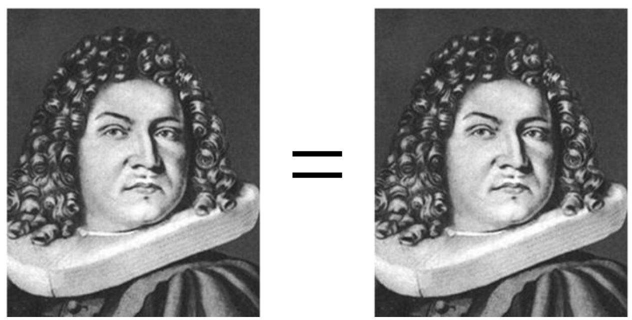
$$\sum_{k=2}^{m} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)] = \frac{B_m}{m!}(m!) = B_m$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1) + f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$0 = B_m - B_m$$

$$B_m = B_m$$ (Trivial identity!)



Example 4: $f(x) = (1-x)B_m(x) \qquad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 (1-x)B_m(x)dx$$

$$= \left.\frac{(1-x)B_{m+1}(x)}{m+1}\right|_0^1 + \frac{1}{m+1}\int_0^1 B_{m+1}(x)dx$$

$$= -\frac{B_{m+1}}{m+1}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[0 + B_m(0)]$$

$$= \frac{1}{2}B_m$$

$$f^{(n)}(x) = \frac{d^n}{dx^n}[(1-x)B_m(x)] = ?$$

Leibniz's Product Rule:

$$f(x) = g(x)h(x)$$

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

$$f''(x) = [g''(x)h'(x) + g'(x)h'(x)] + [g'(x)h'(x) + g(x)h''(x)]$$

$$= g''(x)h'(x) + 2g'(x)h'(x) + g(x)h''(x)$$

$$f'''(x) = g'''(x)h(x) + 3g''(x)h'(x) + 3g'(x)h''(x) + g(x)h'''(x)$$

$$f^{(n)}(x) = \binom{n}{0} g^{(n)}(x)h(x) + \binom{n}{1} g^{(n-1)}(x)h'(x) + \ldots + \binom{n}{n} g(x)h^{(n)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)}(x)h^{(k)}(x)$$

$$f(x) = (1-x)B_m(x)$$

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}[1-x] \cdot B_m^{(n-k)}(x)$$

$$= (1-x)B_m^{(n)}(x) - n \cdot 1 \cdot B_m^{(n-1)}(x)$$

$$= (1-x)\frac{m!}{(m-n)!}B_{m-n}(x) - n\frac{m!}{(m-n+1)!}B_{m-n+1}(x)$$

$$f^{(k-1)}(1) - f^{(k-1)}(0)$$

$$= -(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}(1) - \left[\frac{m!}{(m-k+1)!}B_{m-k+1}(0) \right.$$

$$\left. -(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}(0)\right]$$

$$= -\frac{m!}{(m-k+1)!}B_{m-k+1} + 2(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}\delta_{m-k+1}$$

where

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$\sum_{k=2}^{m+1}\frac{B_k}{k!}\left[f^{(k-1)}(1)-f^{(k-1)}(0)\right]$$

$$=\sum_{k=2}^{m+1}\frac{B_k}{(k)!}[-\frac{m!}{(m-k+1)!}B_{m-k+1}+2(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}\delta_{m-k+1}]$$

$$=\frac{2m}{m+1}B_{m+1}B_1-\frac{1}{(m+1)}\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

EMS:

$$\int_0^1 f(x)dx=\frac{1}{2}[f(1)+f(0)]-\sum_{k=2}^{p}\frac{B_k}{k!}[f^{(k-1)}(1)-f^{(k-1)}(0)]$$

$$-\frac{B_{m+1}}{m+1}=\frac{1}{2}B_m-\frac{2mB_1B_{m+1}}{m+1}+\frac{1}{(m+1)}\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$-B_{m+1}=\frac{(m+1)}{2}B_m+mB_{m+1}+\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$-B_{m+1} = \frac{(m+1)}{2} B_m + mB_{m+1} + \sum_{k=2}^{m+1} \binom{m+1}{k} B_k B_{m-k+1}$$

$$-mB_{m+1} = (m+1)B_m + \left[B_0 B_{m+1} + (m+1)B_1 B_m + \sum_{k=2}^{m+1} \binom{m+1}{k} B_k B_{m-k+1} \right]$$

$$-mB_{m+1} = (m+1)B_m + \sum_{k=0}^{m+1} \binom{m+1}{k} B_k B_{m-k+1}$$

$$\boxed{\sum_{k=0}^{m+1} \binom{m+1}{k} B_k B_{m-k+1} = -(m+1)B_m - mB_{m+1}}$$

Assume $m+1 = 2n > 2$:

$$\sum_{k=0}^{2n} \binom{2n}{k} B_k B_{2n-k} = -(2n+1)B_{2n-1} - (2n-1)B_{2n}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n-1)B_{2n} \quad \text{(recover Euler's formula)}$$

Hypergeometric Bernoulli Polynomials (N = 2)

$$B_0(2,x)=1, \qquad B_1(2,x)=x-\frac{1}{3},$$

$$B_2(2,x)=x^2-\frac{2}{3}x+\frac{1}{18}, \quad B_3(2,x)=x^3-x^2+\frac{1}{6}x+\frac{1}{90}.$$

Appell sequence with zero first moment:

(i) $B_0(2,x)=1$

(ii) $B_n\,'(2,x)=nB_{n-1}(2,x)$

(iii) $\int_0^1(1-x)B_n(2,x)dx=\begin{cases}1/2 & \text{if } n=0\\ 0 & \text{if } n>0\end{cases}$

We define *hypergeometric Bernoulli numbers* by

$$B_n(2)=B_n(2,0)$$

$$B_0(2)=1, \quad B_1(2)=-\frac{1}{3}, \quad B_2(2)=\frac{1}{18}, \quad B_3(2)=\frac{1}{90}$$

Example 5: $f(x) = B_m(2, x) \qquad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 B_m(2, x)dx = \begin{cases} B_m(2) + m/2 & \text{if } m = 1 \\ B_m(2) & \text{if } m \neq 1 \end{cases}$$

Proof:

$$\int_0^1 (1-x)B_{m-1}(2, x)dx = \begin{cases} 1/2 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}$$

$$\left.\frac{(1-x)B_m(2, x)}{m}\right|_0^1 + \frac{1}{m}\int_0^1 B_m(2, x)dx = \begin{cases} 1/2 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}$$

$$-\frac{B_m(2,0)}{m} + \frac{1}{m}\int_0^1 B_m(2, x)dx = \begin{cases} 1/2 & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[B_m(2,1) + B_m(2,0)] = B_m(2) + \frac{m}{2}B_{m-1}(2) \quad (m > 1)$$

Proof:

$$\frac{1}{2}[B_m(2,1)+B_m(2,0)] = \frac{1}{2}[B_m(2,1)-B_m(2,0)]+B_m(2,0)$$

$$= \frac{m}{2}\int_0^1 B_{m-1}(2,x)dx + B_m(2,0)$$

$$= \frac{m}{2}B_{m-1}(2)+B_m(2)$$

$$f^{(k-1)}(1)-f^{(k-1)}(0) = B_m^{(k-1)}(2,1)-B_m^{(k-1)}(2,0)$$

$$= \frac{m!}{(m-k+1)!}\ B_{m-k+1}(2,1)-B_{m-k+1}(2,0)$$

$$= \frac{m!}{(m-k+1)!}(m-k+1)\int_0^1 B_{m-k}(2,x)dx$$

$$= \frac{m!}{(m-k)!}\left[B_{m-k}(2,1)+\frac{m-k}{2}\delta_{m-k-1}\right]$$

$$= \frac{m!}{(m-k)!}B_{m-k}(2)+\frac{m!}{2(m-k-1)!}\delta_{m-k-1}$$

$$\sum_{k=2}^{m}\frac{B_k}{k!}\left[f^{(k-1)}(1)-f^{(k-1)}(0)\right]=\sum_{k=2}^{m}\frac{B_k}{k!}\left[\frac{m!}{(m-k)!}B_{m-k}(2)+\frac{m!}{2(m-k-1)!}\delta_{m-k-1}\right]$$

$$=\frac{B_{m-1}}{(m-1)!}\cdot\frac{m!}{2}+\sum_{k=2}^{m}\frac{m!}{k!(m-k)!}B_kB_{m-k}(2)$$

$$=\frac{mB_{m-1}}{2}+\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)$$

EMS:

$$\int_0^1 f(x)dx=\frac{1}{2}[f(1)+f(0)]-\sum_{k=2}^{p}\frac{B_k}{k!}[f^{(k-1)}(1)-f^{(k-1)}(0)]$$

$$B_m(2)=B_m(2)+\frac{m}{2}B_{m-1}(2)-\frac{mB_{m-1}}{2}-\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)$$

$$B_m(2)+m\left(-\frac{1}{2}\right)B_{m-1}(2)+\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(2)=-\frac{mB_{m-1}}{2}+B_m(2)$$

Theorem:

$$\sum_{k=0}^{m}\binom{m}{k}B_k B_{m-k}(2) = -\frac{mB_{m-1}}{2} + B_m(2)$$

Efficient formula for $B_m(2)$:

$$B_m(2) = B_m + \frac{2}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k}B_k(2)B_{m+1-k}$$

$$= \begin{cases} B_m + \dfrac{2}{m+1}\displaystyle\sum_{j=0}^{\frac{m-2}{2}}\binom{m+1}{2j+1}B_{2j+1}(2)B_{m-2j} & \text{if } m \text{ even} \\ B_m + \dfrac{2}{m+1}\displaystyle\sum_{j=0}^{\frac{m-1}{2}}\binom{m+1}{2j}B_{2j}(2)B_{m+1-2j} & \text{if } m \text{ odd} \end{cases}$$

Interesting Problem: Find a formula for $B_m(2)$ in terms of only Bernoulli numbers.

Further Exploration

Other choices for $f(x)$:

$$f(x) = (1-x)B_m(x)B_n(x) \qquad \text{(Cubic recurrence formula?)}$$

$$f(x) = E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] \quad \text{(Euler polynomials)}$$

$$f(x) = H_n(x) = (-1)^n e^{x^2/2} \frac{d^n(e^{-x^2/2})}{dx^n} \quad \text{(Hermite polynomials)}$$

Generalized Euler-Maclaurin Summation (GEMS)

Appell Sequence: $\{A_n(x)\}$

(i) $A_0(x) = 1$

(ii) $A_n'(2, x) = nA_{n-1}(2, x)$

Repeated Integration by Parts:

$$\int_a^b f(x)dx = \int_a^b A_0(x) f(x)dx$$

$$= [A_1(x) f(x)]_a^b - \int_a^b A_1(x) f'(x)dx$$

$$= [A_1(b) f(b) - A_1(a) f(a)] - \left[\frac{A_2(x) f'(x)}{2!} \right]_a^b$$

$$+ \frac{1}{2!} \int_a^b A_2(x) f''(x)dx$$

$$\int_a^b f(x)dx = [A_1(b)f(b) - A_1(a)f(a)] - \frac{1}{2!}[A_2(b)f'(b) - A_2(a)f'(a)]$$

$$+\left[\frac{A_3(x)f''(x)}{3!}\right]_a^b - \frac{1}{3!}\int_a^b A_3(x)f'''(x)dx$$

GEMS:

$$\int_a^b f(x)dx = \sum_{k=1}^{p}\frac{(-1)^{k-1}}{k!}[A_k(b)f^{(k-1)}(b) - A_k(a)f^{(k-1)}(a)]$$

$$+\frac{(-1)^k}{p!}\int_a^b A_p(x)f^{(p)}(x)dx$$

Choices for $A_k(x)$:

$$A_k(x) = B_k(2, x)$$

$$A(x) = E_n(x)$$

$$A(x) = H_n(x)$$

References

[1] L. Euler, [E212] Institutiones calculi differentialis (1713), Part II, Chapter 5, Translation by David Pengelley, available at The Euler Archive: http://www.math.dartmouth.edu/~euler/

[2] A. Hassen and H. Nguyen, *Hypergeometric Bernoulli Polynomials and Appell Sequences*, to appear in Intern. J. Number Theory.