Paradoxical Euler
or
Integration by Differentiation:

A Synopsis of E236

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E236

Exposition de quelques paradoxes dans le calcul integral
(Explanation of Certain Paradoxes in Integral Calculus)

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English translation by Andrew Fabian (2007).

Student Translations of Euler's Mathematics (STEM) Project

- 7 students
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The First Paradox

EXPOSITION DE QUELQUES PARADOXES DANS LE CALCUL INTÉGRAL

PAR M. EULER.

Premier Paradoxe.

I.

Je me propoſe ici de développer un paradoxe dans le calcul intégral, qui paroitra bien étrange : c'eſt qu'on parvient quelquefois à des équations différentielles, dont il paroit fort difficile de trouver les intégrales par les régles du calcul intégral, & qu'il eſt pourtant aiſé de trouver, non par le moyen de l'intégration, mais plutôt en différentiant encore l'équation propoſée ; de ſorte qu'une différentiation réitérée nous conduiſe dans ces cas à l'intégrale cherchée. C'eſt ſans doute un accident fort ſurprenant, que la différentiation nous puiſſe mener au même but, auquel on eſt accoutumé de parvenir par l'intégration qui eſt une opération entierement oppoſée.

II. Pour mieux faire ſentir l'importance de ce paradoxe, on n'a qu'à ſe ſouvenir, que le calcul intégral renferme la méthode naturelle de trouver les intégrales des quantités différentielles quelconques : & de là il ſemble qu'une équation différentielle étant propoſée, il n'y a d'autre moyen pour arriver à ſon intégrale, que d'en entreprendre l'intégration. Et ſi l'on vouloit, au lieu d'intégrer cette équation, la différentier encore une fois, on devroit croire qu'on s'éloigneroit encore davantage du but propoſé ; attendu qu'on auroit alors une équation différentielle du ſecond degré, qu'il faudroit même deux fois intégrer, avant qu'on parvint aut but propoſé.

III.

EXPLANATION OF CERTAIN PARADOXES IN INTEGRAL CALCULUS

BY MR. EULER

Translation from the French: ANDREW FABIAN

The First Paradox

I.

Here I intend to explain a paradox in integral calculus that will seem rather strange: this is that we sometimes encounter differential equations in which it would seem very difficult to find the integrals by the rules of integral calculus yet are still easily found, not by the method of integration, but rather in differentiating the proposed equation again; so in these cases, a repeated differentiation leads us to the sought integral. This is undoubtedly a very surprising accident, that differentiation can lead us to the same goal, to which we are accustomed to find by integration, which is an entirely opposite operation.

II. To get a better feel for the importance of this paradox, we only have to remember that integral calculus holds the natural method for finding integrals from differential quantities: and from this it seems that for a proposed differential equation, there is no other way to arrive at its integral than to attempt its integration. And if we would, instead of integrating this equation, differentiate it once more, we would need to believe that we would further distance ourselves from the proposed goal; considering that we would then have a differential equation of the second degree, it would need two integrations before we reach the proposed goal.

III. Il doit donc être très ſurprenant, qu'une différentiation réitérée ne nous éloigne non ſeulement davantage de l'intégrale, que nous nous propoſons de chercher, mais qu'elle nous puiſſe même fournir cette intégrale. Ce ſeroit ſans doute un grand avantage, ſi cet accident étoit général, & qu'il eut lieu toujours, puiſqu'alors la recherche des intégrales, qui eſt ſouvent même impoſſible, n'auroit plus la moindre difficulté : mais il ne ſe trouve qu'en quelques cas très particuliers dont je rapporterai quelques exemples : les autres cas demandent toujours la méthode ordinaire d'intégration. Voilà donc quelques problèmes qui ſerviront à éclaircir ce paradoxe.

PROBLEME I.

Le point A *étant donné, trouver la courbe* EM *telle, que la perpendiculaire* AV *tirée du point* A *ſur une tangente quelconque de la courbe* MV, *ſoit partout de la même grandeur.* Fig. 1.

IV. Prenant pour axe une droite quelconque AP, tirée du point donné A, qu'on y tire d'un point quelconque de la courbe cherchée M la perpendiculaire MP, & une autre infiniment proche mp : & qu'on nomme $AP = x$, $PM = y$, & la longueur donnée de la ligne $AV = a$. Soit de plus l'élément de la courbe $Mm = ds$, & ayant tiré $M\pi$ parallele à l'axe AP, on aura $Pp = M\pi = dx$ & $\pi m = dy$; donc $ds = \sqrt{(dx^2 + dy^2)}$. Qu'on baiſſe du point P auſſi ſur la tangente MV la perpendiculaire PS, & ſur celle-cy du point A la perpendiculaire AR, qui ſera parallele à la tangente MV. Maintenant, puiſque les triangles PMS & APR ſont ſemblables au triangle $Mm\pi$, on en tirera : $PS = \frac{M\pi . PM}{Mm} = \frac{ydx}{ds}$ & $PR = \frac{m\pi . AP}{Mm} = \frac{xdy}{ds}$: d'où, à cauſe de $AV = PS - PR$, nous aurons cette équation, $a = \frac{ydx - xdy}{ds}$ ou $ydx - xdy = ads$

$= a$

III. It must therefore be very surprising that a repeated differentiation does not distance us only further from the integral that we proposed to find, but it can even give us this integral. This would undoubtedly be a great advantage, if this accident were general and always held true, since then the study of integrals, which are often impossible, would no longer pose the least difficulty: but it is only found in some very particular cases in which I will relate some examples: the other cases always follow the ordinary method of integration. Therefore, here are some problems that serve to clarify this paradox.

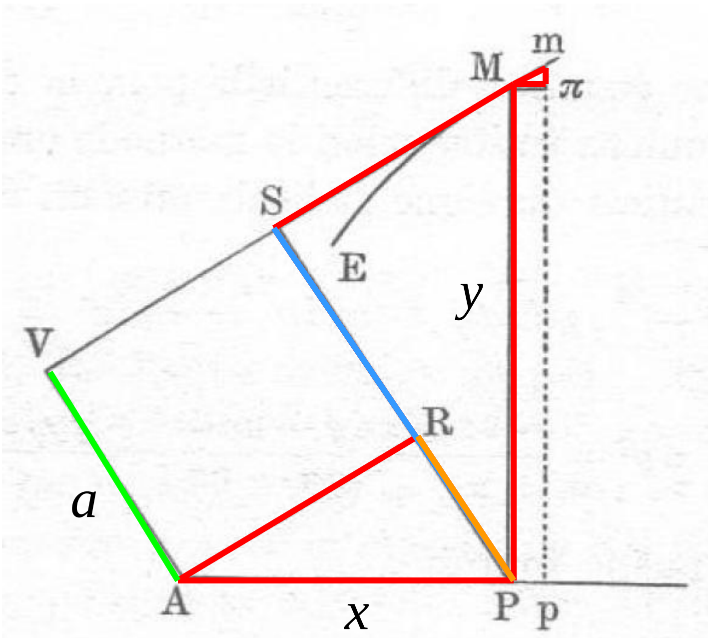
PROBLEM I

Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere. (Fig. 1)

IV. Taking for the axis some straight line *AP* derived from the given point *A*, we derive the perpendicular *MP* there from some point *M* on the sought curve and another infinitely close line *mp*. Also, let us call $AP = x$, $PM = y$, and the given length of the line $AV = a$. Furthermore, let the element of the curve $Mm = ds$, and having derived $M\pi$ parallel to the axis *AP*, we will have $Pp = M\pi = dx$ and $\pi m = dy$; therefore $ds = \sqrt{(dx^2 + dy^2)}$. We extend from the point *P* also onto the tangent *MV* the perpendicular *PS* and onto this line from the point *A* the perpendicular *AR*, which will be parallel to the tangent *MV*. Now, since the triangles *PMS* and *APR* are similar to the triangle $Mm\pi$, we can derive: $PS = \frac{M\pi \cdot PM}{Mm} = \frac{ydx}{ds}$ and $PR = \frac{m\pi \cdot AP}{Mm} = \frac{xdy}{ds}$: from

PROBLEM I (E236)

Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere.



$\Delta Mm\pi, \Delta PMS$, and ΔAPR are similar

$$\frac{PS}{PM} = \frac{M\pi}{Mm} \Rightarrow PS = \frac{PM \cdot M\pi}{Mm} = \frac{ydx}{ds}$$

$$\frac{PR}{AP} = \frac{m\pi}{Mm} \Rightarrow PR = \frac{AP \cdot m\pi}{Mm} = \frac{xdy}{ds}$$

$x = AP$ $\quad dx = Pp = M\pi$ $\quad ds = Mm = \sqrt{dx^2 + dy^2}$

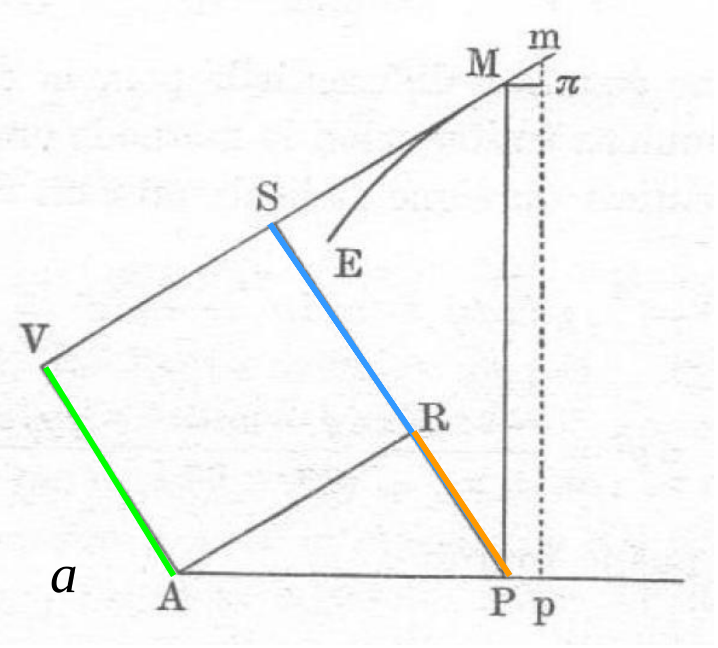
$y = PM$ $\quad dy = m\pi$ $\quad AV = a$ ("Tangential" distance)

$$a = AV = PS - PR$$

$$= \frac{ydx}{ds} - \frac{xdy}{ds}$$

$$ydx - xdy = ads$$

$$y - x\frac{dy}{dx} = a\frac{ds}{dx}$$



Euler's Differential Equation:

$$y - x\frac{dy}{dx} = a\frac{\sqrt{dx^2 + dy^2}}{dx}$$

Note: If the solution curve is assumed to lie below *x*-axis, then Euler's DE becomes

$$(-y) - x\frac{-dy}{dx} = a\frac{\sqrt{(dx)^2 + (-dy)^2}}{dx} \Rightarrow y - x\frac{dy}{dx} = -a\frac{\sqrt{dx^2 + dy^2}}{dx}$$

Ordinary Method of Integration

Euler's differential equation (differential form):

$$y\,dx - x\,dy = a\sqrt{dx^2 + dy^2}$$

1. Square both sides:

$$y^2 dx^2 - 2xy\,dx\,dy + x^2 dy^2 = a^2 dx^2 + a^2 dy^2$$

2. Solve for *dy* (Euler ignores negative solution):

$$dy = \frac{-xy\,dx + a\,dx\sqrt{x^2 + y^2 - a^2}}{a^2 - x^2}$$

3. Rewrite:

$$a^2 dy - x^2 dy + xy\,dx = a\,dx\sqrt{x^2 + y^2 - a^2}$$

$$a^2\,dy - x^2\,dy + xy\,dx = a\,dx\sqrt{x^2 + y^2 - a^2}$$

4. *u*-substitution:

$$y = u\sqrt{a^2 - x^2}; \quad dy = du\sqrt{a^2 - x^2} - \frac{ux\,dx}{\sqrt{a^2 - x^2}}$$

LHS:

$$(a^2 - x^2)dy + xy\,dx \Rightarrow (a^2 - x^2)\left(du\sqrt{a^2 - x^2}\right) - (a^2 - x^2)\left(\frac{ux\,dx}{\sqrt{a^2 - x^2}}\right) + ux\,dx\sqrt{a^2 - x^2}$$

$$\Rightarrow du(a^2 - x^2)^{3/2} - \cancel{ux\,dx\sqrt{a^2 - x^2}} + \cancel{ux\,dx\sqrt{a^2 - x^2}}$$

RHS:

$$a\,dx\sqrt{x^2 + y^2 - a^2} \Rightarrow a\,dx\sqrt{(a^2 - x^2)(u^2 - 1)}$$

Simplifies to:

$$du(a^2 - x^2)^{3/2} = a\,dx\sqrt{(a^2 - x^2)(u^2 - 1)}$$

$$\therefore (a^2 - x^2)du = a\,dx\sqrt{u^2 - 1}$$

Case I: $u^2 = 1$

$$(a^2 - x^2)du = a\,dx\sqrt{u^2 - 1} \text{ (true)}$$

Thus:

$$y = u\sqrt{a^2 - x^2} = \pm\sqrt{a^2 - x^2}$$

$$\therefore \quad x^2 + y^2 = a^2 \quad \text{(Circle)}$$

Case II: $u^2 \neq 1$

$$\int \frac{du}{\sqrt{u^2 - 1}} = a \int \frac{dx}{a^2 - x^2}$$

$$\log\left(u + \sqrt{u^2 - 1}\right) = \log\left(n \left(\frac{a + x}{a - x} \right)^{1/2} \right)$$

$$u+\sqrt{u^2-1}=n\left(\frac{a+x}{a-x}\right)^{1/2}$$

$$n^2\left(\frac{a+x}{a-x}\right)-2u(u+\sqrt{u^2-1})=-1$$

$$n^2\left(\frac{a+x}{a-x}\right)-2un\left(\frac{a+x}{a-x}\right)^{1/2}=-1$$

$$\frac{-n}{2}\left(\frac{a+x}{a-x}\right)^{1/2}+u=\frac{1}{2n}\left(\frac{a+x}{a-x}\right)^{-1/2}$$

$$\frac{-n}{2}\left(\frac{a+x}{a-x}\right)^{1/2}+\frac{y}{\sqrt{(a+x)(a-x)}}=\frac{1}{2n}\left(\frac{a+x}{a-x}\right)^{-1/2}$$

$$\frac{-n}{2}(a+x)+y=\frac{1}{2n}(a-x)$$

$$\boxed{y=\frac{(n^2-1)}{2n}x+\frac{(n^2+1)}{2n}a} \qquad \text{(Line)}$$

Integrating by Differentiating

Euler's Differential Equation:

$$y\,dx - x\,dy = a\sqrt{dx^2 + dy^2}$$

1. Rewrite in terms of $p = \dfrac{dy}{dx}$:

$$y - xp = a\sqrt{1+p^2}$$

2. Differentiate: $\dfrac{dy}{dx} - \left(1\cdot p + x\dfrac{dp}{dx}\right) = \dfrac{ap}{\sqrt{1+p^2}}\cdot\dfrac{dp}{dx}$

$$\cancel{p} - \cancel{p} - x\frac{\cancel{dp}}{\cancel{dx}} = \frac{ap}{\sqrt{1+p^2}}\cdot\frac{\cancel{dp}}{\cancel{dx}}$$

Case I: $\dfrac{dp}{dx} \neq 0$

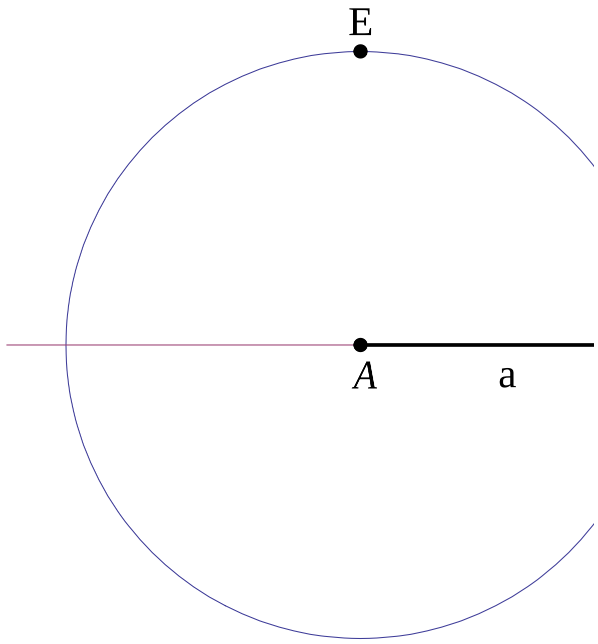
$$x = \boxed{-\frac{ap}{\sqrt{1+p^2}}} \qquad y = a\sqrt{1+p^2} + xp = \boxed{\frac{a}{\sqrt{1+p^2}}}$$

Eliminate the parameter:

$$x^2 + y^2 = \left(-\frac{ap}{\sqrt{1+p^2}} \right)^2 + \left(\frac{a}{\sqrt{1+p^2}} \right)^2 = \frac{a^2 p^2 + a^2}{1+p^2} = a^2$$

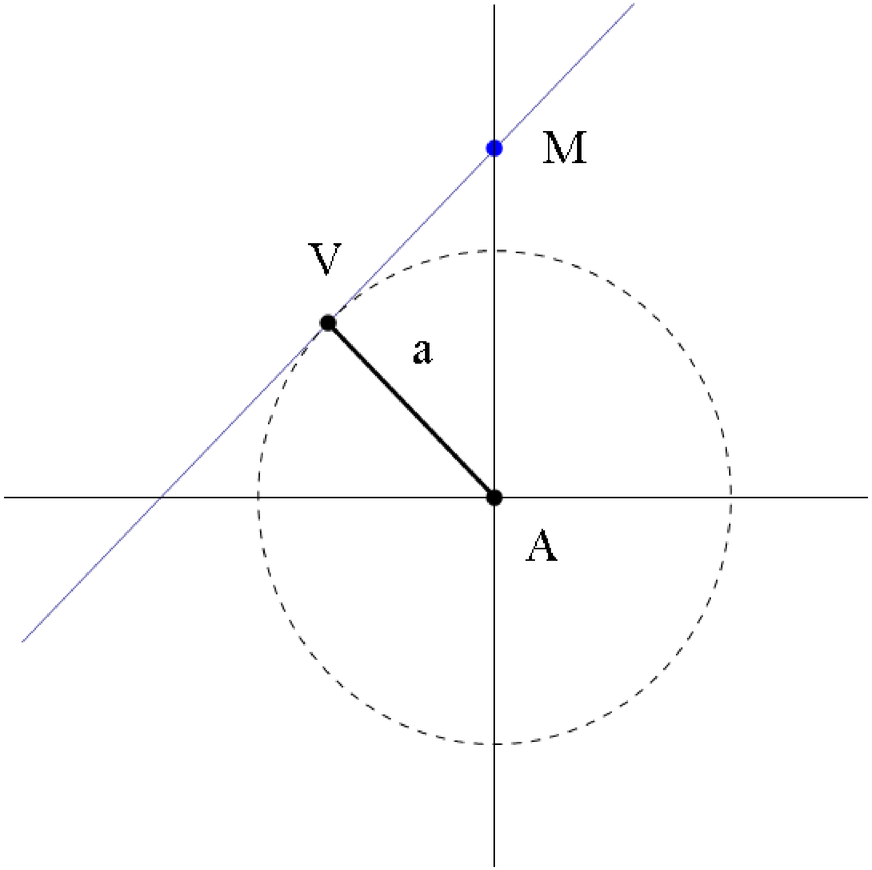
Solution to Case I:

$$\therefore x^2 + y^2 = a^2 \qquad \text{(Circle)}$$



Case II: $\dfrac{dp}{dx} = 0$ Thus p = constant = n

$$y - xp = a\sqrt{1 + p^2} \qquad \therefore\ y = nx + a\sqrt{1 + n^2} \qquad \text{(Tangent line)}$$



Higher-Order Examples

Cubic:

$$y\,dx - x\,dy = a\sqrt[3]{dx^3 + dy^3}$$

$$\Rightarrow y - xp = a\sqrt[3]{1+p^3} \quad \left(p = \frac{dy}{dx}\right)$$

Case I: $\dfrac{dp}{dx} \neq 0$

$$x = -\frac{ap^2}{\sqrt[3]{(1+p^3)^2}} \qquad y = \frac{a}{\sqrt[3]{(1+p^3)^2}}$$

Case II: $\dfrac{dp}{dx} = 0$ Thus p = constant = n

$$y - xp = a\sqrt[3]{1+p^3} \qquad \therefore y = nx + a\sqrt[3]{1+n^3} \qquad \text{(Tangent line)}$$

Arbitrary order:

$$ydx - xdy = a\sqrt[n]{(\alpha dx^n + \beta dx^{n-\nu} dy^\nu + \gamma dx^{n-\mu} dy^\mu + etc.)}$$

$$\Rightarrow y = px + a\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)}$$

Case I: $\dfrac{dp}{dx} \neq 0$

$$x = \frac{-\nu a\beta p^{\nu-1} - \mu a\gamma p^{\mu-1} - etc.}{n\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)^{n-1}}}$$

$$y = \frac{na\alpha + (n-\nu)a\beta p^\nu + (n-\mu)a\gamma p^\mu + etc.}{n\sqrt[n]{(\alpha + \beta p^\nu + \gamma p^\mu + etc.)^{n-1}}}$$

Case II: $\dfrac{dp}{dx} = 0$ Thus p = constant = m

$$y = mx + a\sqrt[n]{(\alpha + \beta m^\nu + \gamma m^\mu + etc.)}$$

General Solution of Euler's DE

$$y - xp = F(p) \qquad \left(p = \frac{dy}{dx} \right)$$

Integrating by Differentiating:

$$\frac{dy}{dx} - \left(1 \cdot p + x\frac{dp}{dx} \right) = F'(p) \cdot \frac{dp}{dx}$$

$$\cancel{p} - \cancel{p} - x\frac{dp}{\cancel{dx}} = F'(p) \cdot \frac{dp}{\cancel{dx}}$$

Case I: $\frac{dp}{dx} \neq 0$

$$x = \boxed{-F'(p)}$$

$$y = F(p) + xp = \boxed{F(p) - pF'(p)}$$

(Parametric solution!)

Case II: $\frac{dp}{dx} = 0$ Thus p = constant

$$y = xp + F(p)$$

(Tangent lines to solution in Case I)

PROBLEM II (E236)

On the axis AB, find the curve AMB such that having derived from one of its points M the tangent TMV, it intersects the two lines AE and BF, derived perpendicularly to the axis AB at the two given points A and B, so that the "rectangle" formed by the lines AT and BV is the same size everywhere.

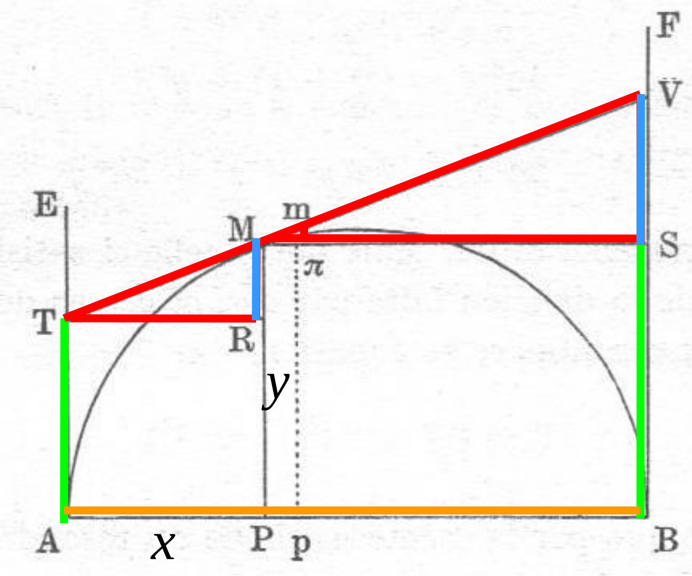


Fig. 2

$\Delta Mm\pi, \Delta PMS,$ and ΔAPR are similar

$$RM = \frac{TR \cdot m\pi}{M\pi} = \frac{x\,dy}{dx}$$

$$SV = \frac{MS \cdot m\pi}{M\pi} = \frac{(2a-x)\,dy}{dx}$$

$x = AP$ $\quad dx = Pp = M\pi$ $\quad ds = Mm = \sqrt{dx^2 + dy^2}$ $\quad AB = 2a$

$y = PM$ $\quad dy = m\pi$ $\quad AT \cdot BV = c^2$ ("Tangential area")

$$AT = PM - RM = y - \frac{x\,dy}{dx}$$

$$BV = BS + SV = y + \frac{(2a - x)\,dy}{dx}$$

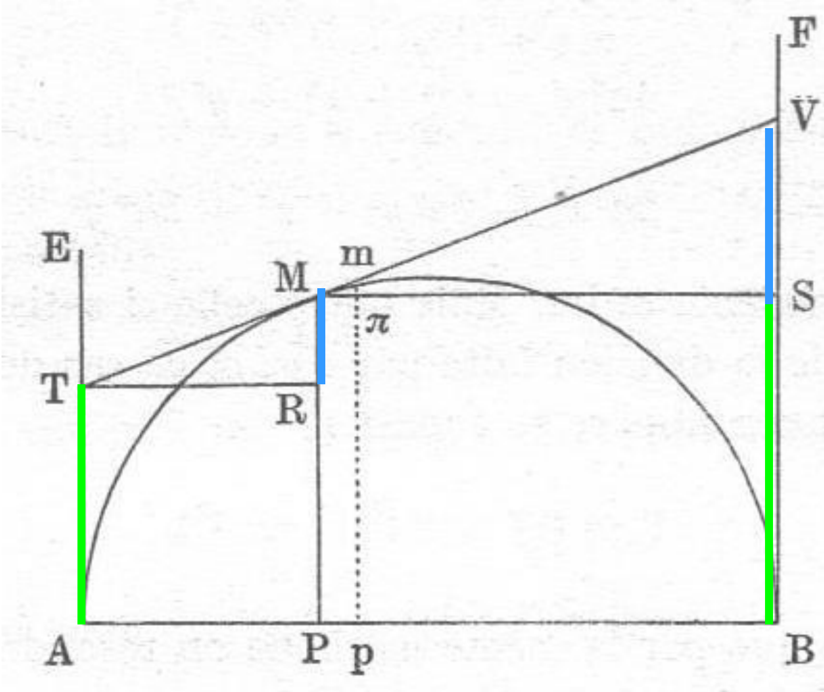


Fig. 2

Euler's Differential Equation

$$AT \cdot BV = \left(y - \frac{xdy}{dx} \right)\left(y - \frac{x\,dy}{dx} + \frac{2ady}{dx} \right) = c^2$$

$$\Rightarrow y - xp = -ap + \sqrt{c^2 + a^2 p^2} = F(p) \qquad \left(p = \frac{dy}{dx} \right)$$

Integrating by Differentiating

Case I: $\frac{dp}{dx} \neq 0$

$$x = a - \frac{a^2 p}{\sqrt{c^2 + a^2 p^2}} \quad \Rightarrow \frac{(x-a)^2}{a^2} = \frac{a^2 p^2}{c^2 + a^2 p^2}$$

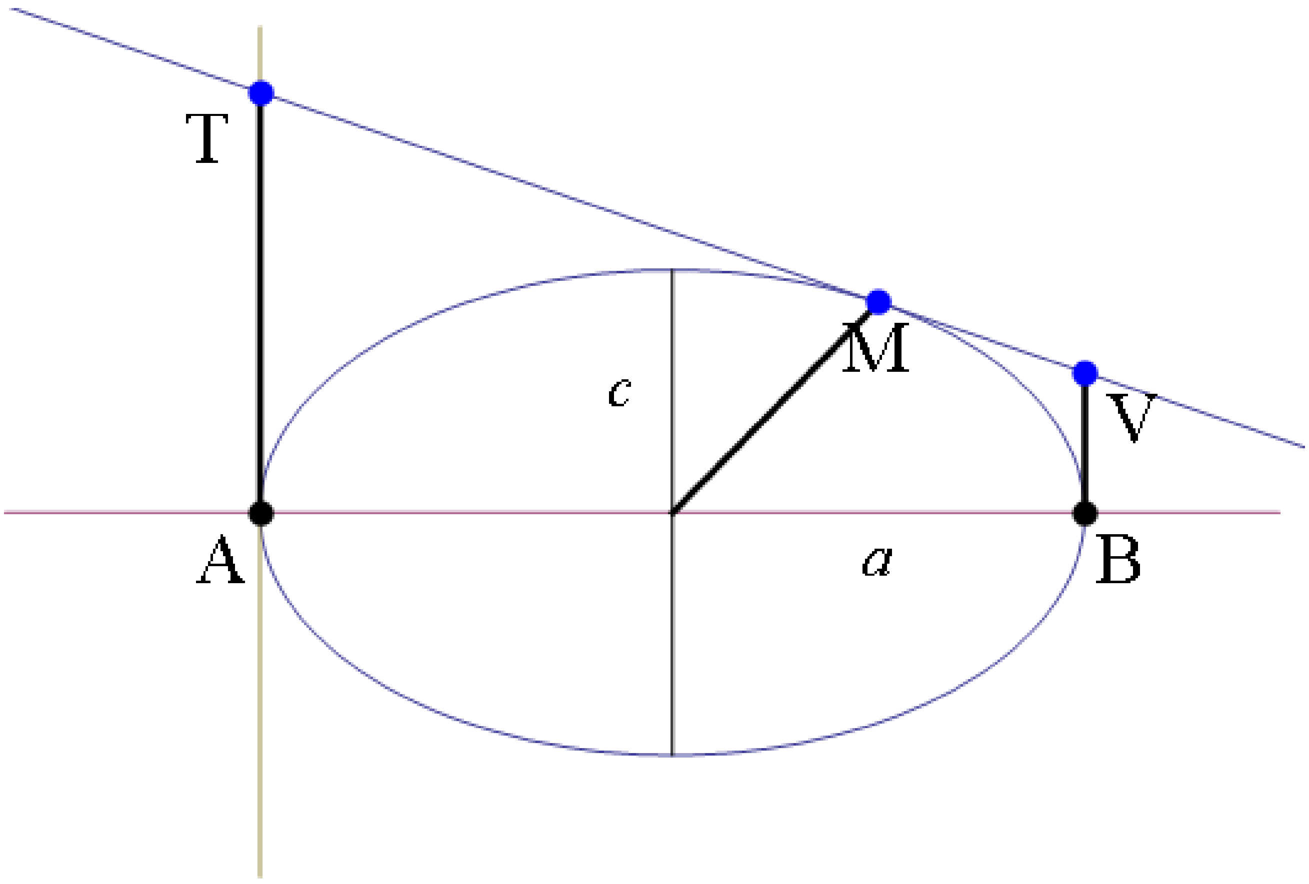
$$y = \frac{c^2}{\sqrt{c^2 + a^2 p^2}} \quad \Rightarrow \frac{y^2}{c^2} = \frac{c^2}{c^2 + a^2 p^2}$$

Eliminating the parameter:

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{c^2} = \frac{c^2 + a^2 p^2}{c^2 + a^2 p^2} = 1$$

Solution to Case I:

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{c^2}=1 \qquad \text{(Ellipse)}$$



Case II: $\frac{dp}{dx} = 0$ Thus p = constant = n

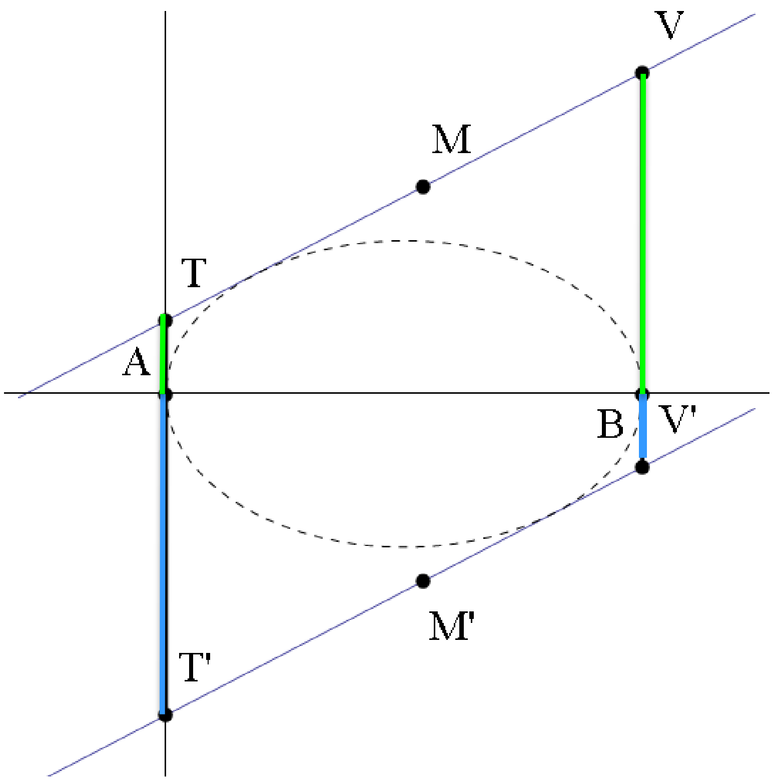
$$y - xp = -ap \pm \sqrt{c^2 + a^2 p^2}$$

$$\therefore y = -n(a - x) \pm \sqrt{c^2 + n^2 a^2}$$ (Tangent line)

$$\therefore AT = -na \pm \sqrt{c^2 + n^2 a^2}$$

$$\therefore BV = na \pm \sqrt{c^2 + n^2 a^2}$$

$$\therefore AT \cdot BV = c^2$$



PROBLEM III (E236)

Given two points A and C, find the curve EM such that if we derive some tangent MV, which the perpendicular AV is directed towards from the first point A, and we join the straight line CV to V from the other point C, this line CV is the same size everywhere.

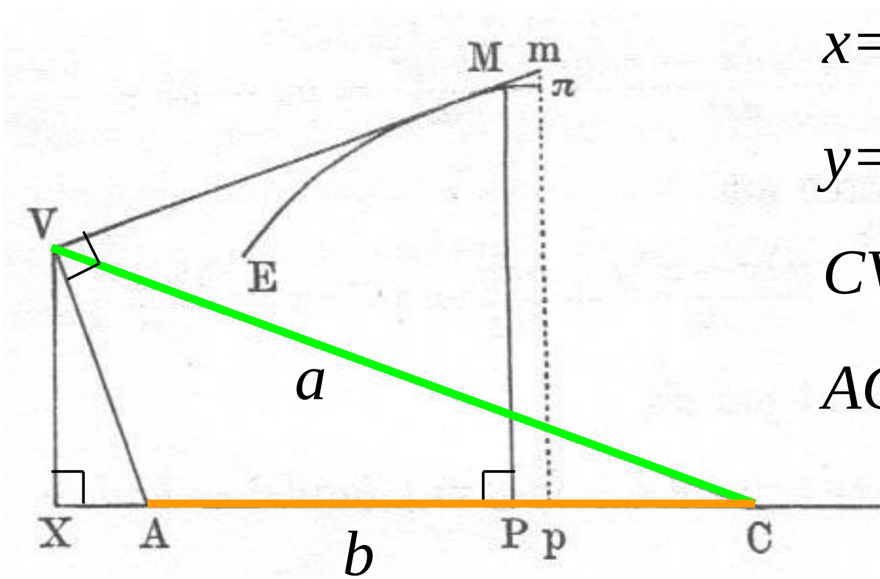


Fig. 3

$x = AP$

$y = PM$

$CV = a$

$AC = b$

PROBLEM IV (E236)

Given two points A and B, find the curve EM such that having derived some tangent VMX, if the perpendiculars AV and BX are directed towards it from the points A and B, the rectangle of these lines is the same size everywhere.

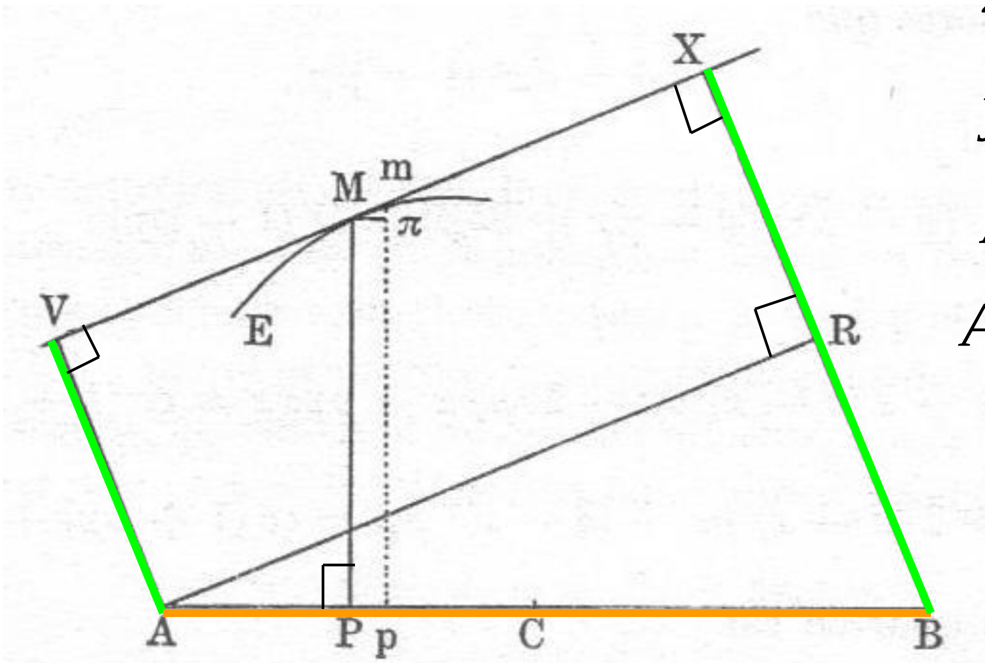


Fig. 4

$x = AP$

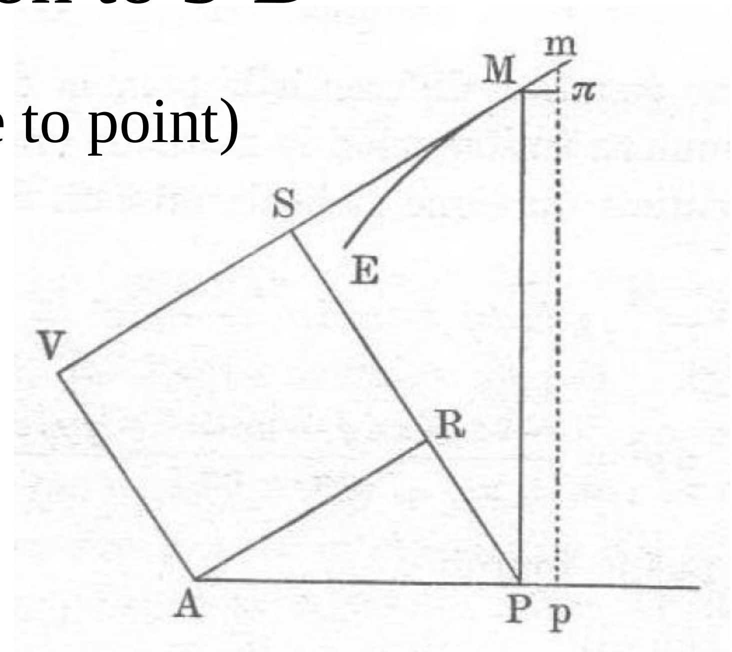
$y = PM$

$AB = 2b$

$AV \cdot BX = c^2$

Generalization to 3-D

2-D: (Distance from curve to point)



$$\langle x, y\rangle \bullet \frac{\langle -dy, dx\rangle}{\sqrt{dx^2+dy^2}} = a$$

$$y\,dx - x\,dy = a\sqrt{dx^2+dy^2}$$

$$y - x\frac{dy}{dx} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

3-D: (Distance from surface to point)

$$z\,dx\,dy - x\,dy\,dz - y\,dx\,dz = a\sqrt{dx^2dy^2+dx^2dz^2+dy^2dz^2}$$

$$z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = a\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

The Second Paradox

Second Paradoxe.

XXXI.

Le ſecond paradoxe, que je m'en vai étaler, n'eſt pas moins ſurprenant, puisqu'il eſt auſſi contraire aux idées communes du calcul intégral. On s'imagine ordinairement, qu'ayant une équation différentielle quelconque, on n'ait qu'à chercher ſon intégrale, & à lui rendre toute ſon étendue en y ajoutant une conſtante indéfinie, pour avoir tous les cas, qui ſont compris dans l'équation différentielle. Ou bien, lorsque cette équation différentielle eſt le réſultat d'une ſolution d'un problème, on ne doute pas que l'équation intégrale, qu'on en trouve par les régles ordinaires, ne renferme toutes les ſolutions poſſibles du problème : cela s'entend, lorsqu'on n'aura pas négligé l'addition de conſtante, que toute intégration exige.

XXXII. Cependant il y a des cas, où l'intégration ordinaire nous conduit à une équation finie, qui ne renferme pas tout ce qui étoit contenu dans l'équation différentielle propoſée ; quand même on ne néglige pas la conſtante mentionnée. Cela doit paroitre d'autant plus paradoxe, plus on eſt accoutumé d'être convaincu de la juſteſſe de l'idée expliquée dans l'article précédent. Car ſi l'équation intégrale, qu'on aura trouvée après toutes les précautions preſcrites, n'épuiſe pas l'étendue de l'équation différentielle ; le problème admettra des ſolutions, que l'intégration ne fournira point, & partant on arrivera à une ſolution défectueuſe, ce qui ſemble ſans doute renverſer les principes ordinaires du calcul intégral.

The Second Paradox

XXXI.

The second paradox that I will put forth is no less surprising, since it is also contrary to the common ideas of integral calculus. We usually imagine that having some differential equation, we only need to find its integral and to render it in its full extent by adding to it an undefined constant to have all the cases that are comprised in the differential equation. Or when this differential equation is resultant from a solution of a problem, we have no doubt that the integral equation found by the ordinary rules contains every possible solution of the problem: this is understood when we have not neglected the addition of a constant that all integration demands.

XXXII. However, there are cases where ordinary integration gives us a finite equation that does not contain all that would be contained in the proposed differential equation, still not neglecting the aforementioned constant. This would seem much more paradoxical since we are accustomed to being convinced of the accuracy of the idea explained in the previous paragraph. Because if the integral equation, which we will have found after all prescribed precautions, does not exhaust the extent of the differential equation, the problem will allow solutions that the integration will not produce, and hence we will arrive at a defective solution that undoubtedly seems to upset the ordinary principles of integral calculus.

References

Originally published in *Memoires de l'academie des sciences de Berlin* 12, 1758, pp. 300-321; *Opera Omnia*: Series 1, Volume 22, pp. 214 – 236. (Available at The Euler Archive: http://www.math.dartmouth.edu/~euler/)

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