Foot to the Pedal:
Constant Pedal Curves and Surfaces

Andrew Fabian
Hieu D. Nguyen

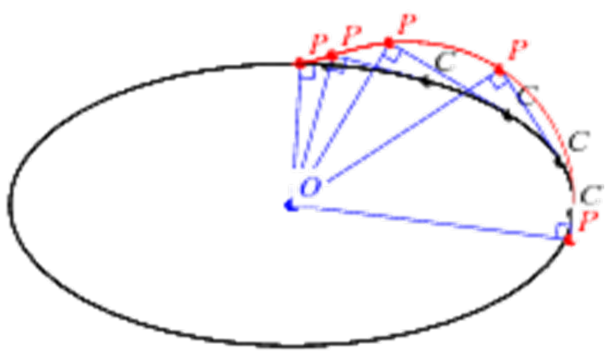
Rowan University

Joint Math Meetings

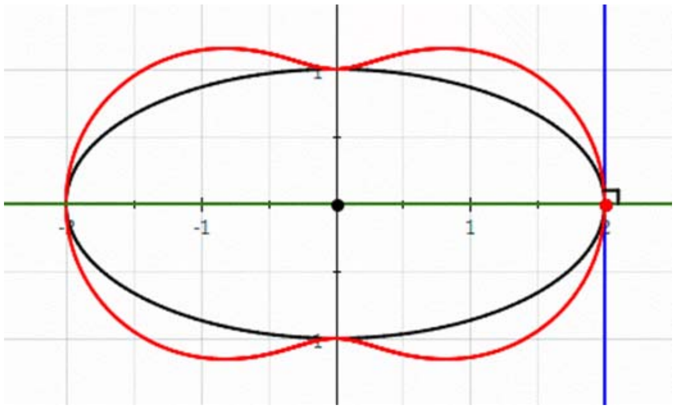
January 13, 2010

Pedal Curves

The *pedal* of a curve *c* with respect to a point *O* (origin) is the locus of the foot of the perpendicular from *O* to the tangent of the curve.



http://mathworld.wolfram.com/PedalCurve.html



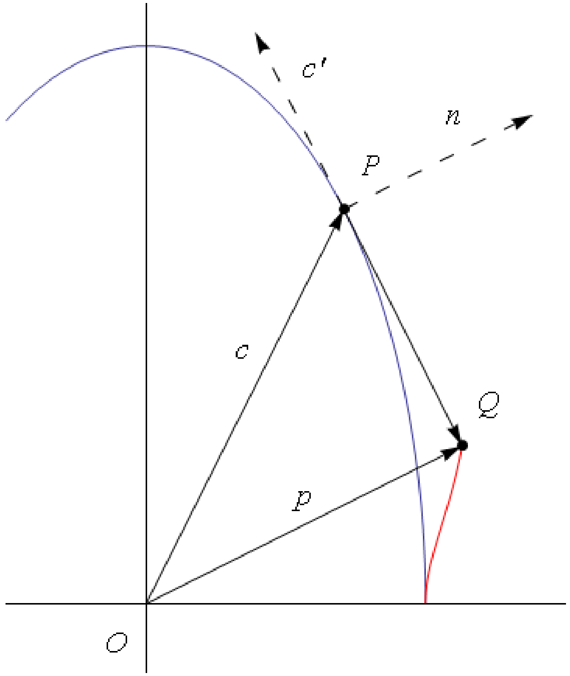
http://en.wikipedia.org/wiki/Pedal_curve

Formula for Pedal Curves

$$c(t) = (x(t), y(t))$$

Tangent vector: $c'(t) = (x'(t), y'(t))$

Normal vector: $n(t) = \dfrac{(y'(t), -x'(t))}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$



Let us denote the pedal of c by p.
Then p is given by the formula

$$p(t) = (c(t) \bullet n(t))n(t)$$

$$= \left(\frac{[x(t)y'(t) - x'(t)y(t)]y'(t)}{[x'(t)]^2 + [y'(t)]^2}, \frac{[x'(t)y(t) - x(t)y'(t)]x'(t)}{[x'(t)]^2 + [y'(t)]^2} \right)$$

Inverse Problem (much more difficult): $c(t) \rightleftarrows p(t)$

Leonard Euler’s E236 Paper

Exposition de quelques paradoxes dans le calcul integral (Explanation of Certain Paradoxes in Integral Calculus)

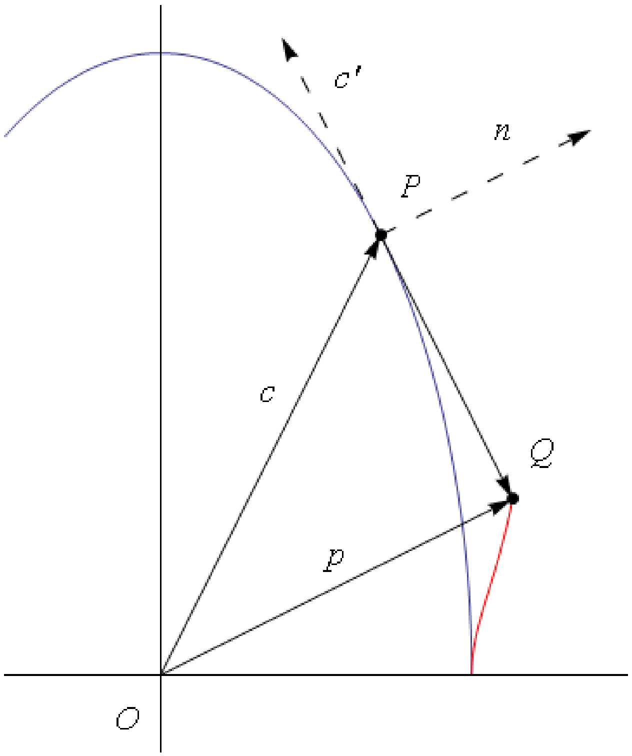
Originally published in *Memoires de l'academie des sciences de Berlin* 12, 1758, pp. 300-321; *Opera Omnia*: Series 1, Volume 22, pp. 214 – 236
(Available at The Euler Archive: http://www.math.dartmouth.edu/~euler/)

English translation by Andrew Fabian (2007).
(Available at The Euler Archive: http://www.math.dartmouth.edu/~euler/)

Curves With Constant Pedal (E236)

Determine a curve c whose pedal has constant distance a from the origin

$$\left|p(t)\right| = \frac{-x(t)y'(t)+y(t)x'(t)}{\sqrt{[x'(t)]^2+[y'(t)]^2}} = a$$



Euler's Differential Equation:

$$y(t)x'(t)-x(t)y'(t) = a\sqrt{[x'(t)]^2+[y'(t)]^2}$$

$$y - x\frac{dy}{dx} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$y - xp = a\sqrt{1+p^2} \quad \left(p = \frac{dy}{dx}\right)$$

Integrating by Differentiating (Euler)

$$y - xp = a\sqrt{1+p^2} \qquad \left(p = \frac{dy}{dx} \right)$$

Differentiate:

$$\frac{dy}{dx} - \left(1 \cdot p + x\frac{dp}{dx} \right) = \frac{ap}{\sqrt{1+p^2}} \cdot \frac{dp}{dx}$$

$$-x\frac{dp}{dx} = \frac{ap}{\sqrt{1+p^2}} \cdot \frac{dp}{dx}$$

Case I: $\dfrac{dp}{dx} \neq 0$

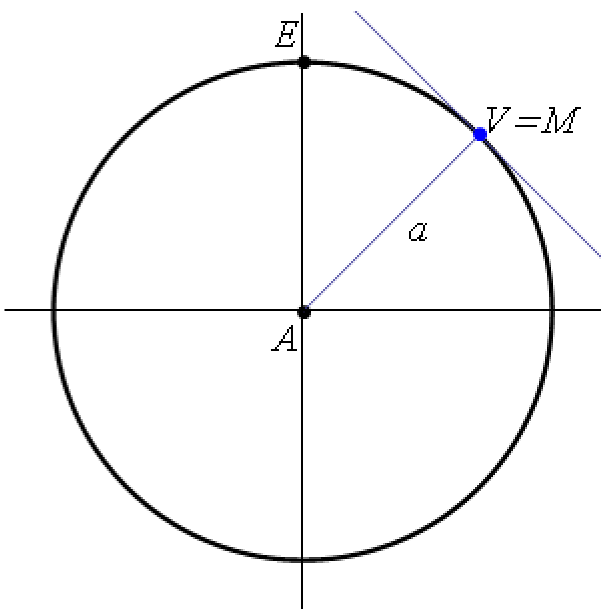
$$x = \boxed{-\frac{ap}{\sqrt{1+p^2}}} \qquad y = a\sqrt{1+p^2} + xp = \boxed{\frac{a}{\sqrt{1+p^2}}}$$

Eliminate the parameter:

$$x^2 + y^2 = \left(-\frac{ap}{\sqrt{1+p^2}} \right)^2 + \left(\frac{a}{\sqrt{1+p^2}} \right)^2 = \frac{a^2 p^2 + a^2}{1+p^2} = a^2$$

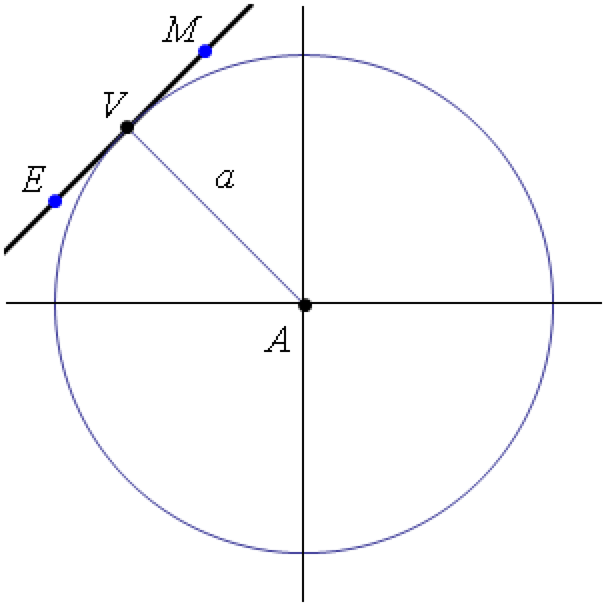
Solution to Case I:

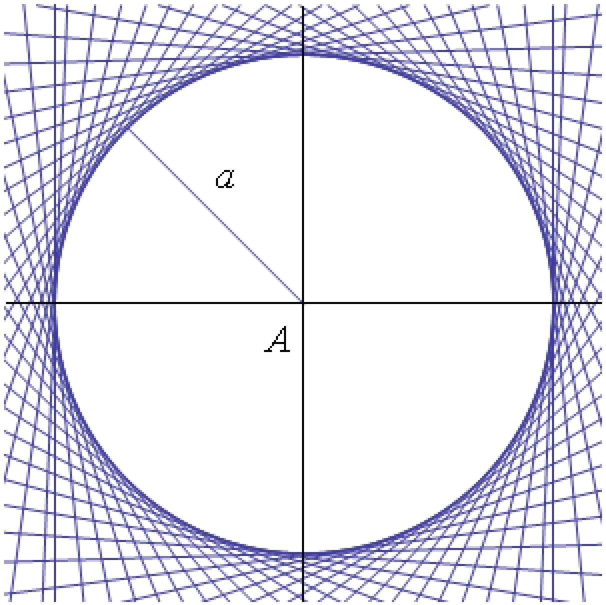
$$\therefore x^2 + y^2 = a^2 \qquad (c = p \text{ is a circle})$$



Case II: $\frac{dp}{dx}=0$ Thus p = constant = n

$$y - xp = a\sqrt{1+p^2} \qquad \therefore y = nx + a\sqrt{1+n^2}$$ (c is a tangent line; p is a point)





Differential Geometry of Plane Curves

Arc length parameter:

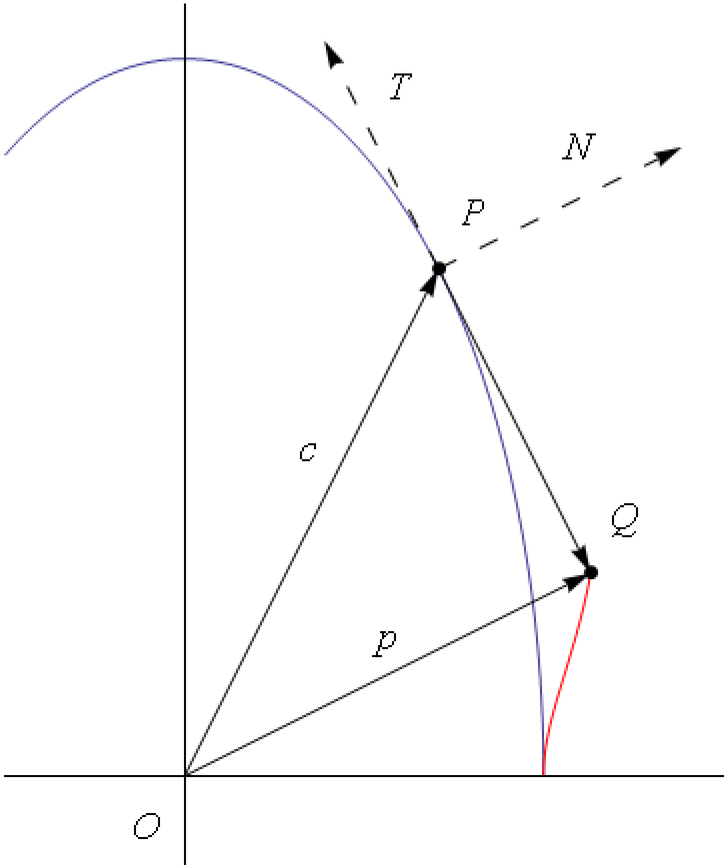
$s(t) = \int_{t_0}^{t} \left| c'(\tilde{t}) \right| d\tilde{t}$ (arc length)

$T(s) = c'(s)$ (unit tangent vector)

$N(t) = \dfrac{T'(s)}{\left| T'(s) \right|}$ (unit normal vector)

$T \perp N$

$\kappa = \left| T'(s) \right|$ (curvature)



Frenet formulas:

$T'(s) = \kappa N(s)$

$N'(s) = -\kappa T(s)$

Solution by Differential Geometry

Curves with constant pedal

$$\left|p(s)\right| = c(s)\bullet N(s) = a$$

Differentiate: $c'(s)\bullet N(s) + c(s)\bullet N'(s) = 0$

$$\Rightarrow \quad 0 + c(s)\bullet N'(s) = 0 \qquad (T \perp N)$$

$$\Rightarrow \quad c(s)\bullet(-\kappa T(s)) = 0$$

$$\Rightarrow \quad \kappa c(s)\bullet c'(s) = 0$$

Case I: $\kappa = 0 \Rightarrow c(s)$ is a line

Case II: $\kappa \neq 0 \Rightarrow c(s)\bullet c'(s) = 0$

$\Rightarrow c(s)\bullet c(s) = k$ (constant)

$\therefore \left|c(s)\right| = \sqrt{k} = a$ (circle)

Pedal Surfaces

The *pedal* of a surface *M* with respect to a point *O* (origin) is the locus of the foot of the perpendicular from *O* to the tangent plane of the surface.

Let us denote the pedal of *M* by *P*. If *M* is described by $z = f(x,y)$, then *P* is given by the formula

$$p(x,y) = (z \bullet \mathbf{n})\mathbf{n}$$

where **n** is the unit normal vector

$$\mathbf{n} = \frac{\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Surfaces With Constant Pedal

Determine a surface M whose pedal has constant distance k from the origin

We will call a surface *M* with constant pedal a *tangentially equidistant* (TED) surface

$$\left|p(x,y)\right| = k = \frac{-x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} + z}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}$$

Thus our surface *S* satisfies

$$z - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = a\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Differential Geometry of TED Surfaces

TED surfaces (with constant pedal)

$$M = \mathbf{x}(u,v) : D \subset \mathbb{R}^2 \to \mathbb{R}^3$$

$$\mathbf{x}(u,v) \bullet \mathbf{n}(u,v) = k$$

Differentiate:

$$\mathbf{x}_u \bullet \mathbf{n} + \mathbf{x} \bullet \mathbf{n}_u = 0 + \mathbf{x} \bullet \mathbf{n}_u = 0$$

$$\mathbf{x}_v \bullet \mathbf{n} + \mathbf{x} \bullet \mathbf{n}_v = 0 + \mathbf{x} \bullet \mathbf{n}_v = 0$$

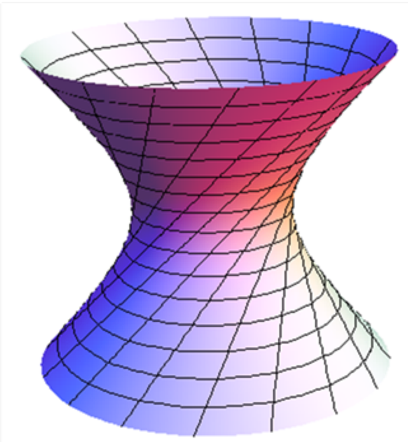
Case I: $\{\mathbf{n}_u, \mathbf{n}_v\}$ are linearly independent on M

$$\operatorname{span}\{\mathbf{n}_u, \mathbf{n}_v\} = T_p M \Rightarrow \begin{cases} \mathbf{x} \bullet \mathbf{x}_u = 0 \\ \mathbf{x} \bullet \mathbf{x}_v = 0 \end{cases} \Rightarrow \mathbf{x} \bullet \mathbf{x} = k \text{ (circle)}$$

Case II: $\{\mathbf{n}_u, \mathbf{n}_v\}$ are NOT linearly independent on M

Gauss curvature $K = 0$ on M (developable) $\Rightarrow M$ is a ruled surface

Ruled TED Surfaces

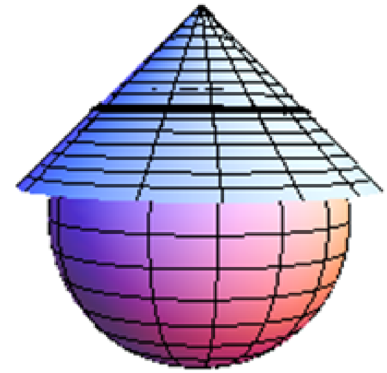


$$\mathbf{x}: D \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$$

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u)$$

$\beta(u)$ - directrix

$\delta(u)$ - ruling



Cone

Observation: Let M be a ruled surface whose directrix β is a curve that lies on the sphere $S^2(k)$ of radius k. Then M is a TED surface with constant pedal β if and only if the tangent planes of M and $S^2(k)$ agree on β.

$$\mathbf{x}_u(u,v) = \beta'(u) + v\delta'(u)$$

$$\mathbf{x}_v(u,v) = \delta(u)$$

Lemma: Assume $\beta(u) \subset S^2(k)$ and $\delta(u) \in T_{\beta(u)}S^2(k)$.
Then $T_{\mathbf{x}(u,v)}M = T_{\beta(u)}S^2(k)$ if and only if $\beta'(u), \delta(u), \delta'(u)$ are coplanar.

Theorem: Let M be a ruled surface having a coordinate patch of the form

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u)$$

where $\beta(u) \subset S^2(k)$. If $\beta'(u), \delta(u), \delta'(u)$ are coplanar, then M is a developable ruled TED surface with constant pedal β.

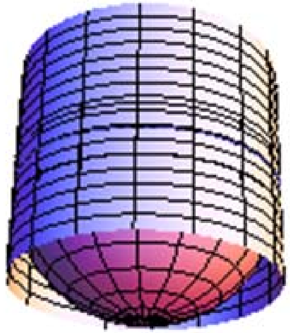
Explicit Construction of TED surfaces:

Lemma: If β is a regular spherical curve and

$$\delta(u) = \beta(u) \times \beta'(u) \in T_{\beta(u)}S^2(k),$$

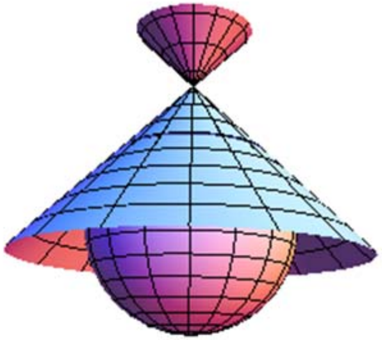
then β, δ, and δ' are coplanar.

Example 1: (Equator)



$$\beta(u) = (\cos u, \sin u, 0)$$
$$\delta(u) = \beta(u) \times \beta'(u) = (0,0,1)$$
$$\mathbf{x}(u,v) = \beta(u) + v\delta(u) = (\cos u, \sin u, v)$$

Example 2: (Latitude circle)



$$\beta(u) = \frac{1}{\sqrt{2}}(\cos u, \sin u, 1)$$
$$\delta(u) = \beta(u) \times \beta'(u) = \frac{1}{2}(-\cos u, -\sin u, 1)$$
$$\mathbf{x}(u,v) = \beta(u) + v\delta(u) = \frac{1}{2}((\sqrt{2}-v)\cos u, (\sqrt{2}-v)\sin u, \sqrt{2}+v)$$

Example 3: (Figure-8)

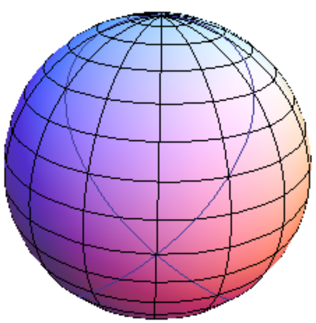
$$\beta(u) = (\cos u \sin u, \sin^2 u, \cos u)$$

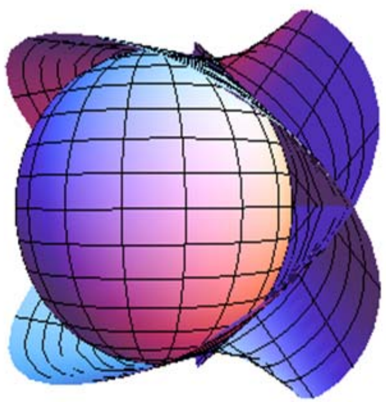
$$\delta(u) = \beta(u) \times \beta'(u) = (-\frac{1}{2}(3+\cos 2u)\sin u, \cos^3 u, \sin^2 u)$$

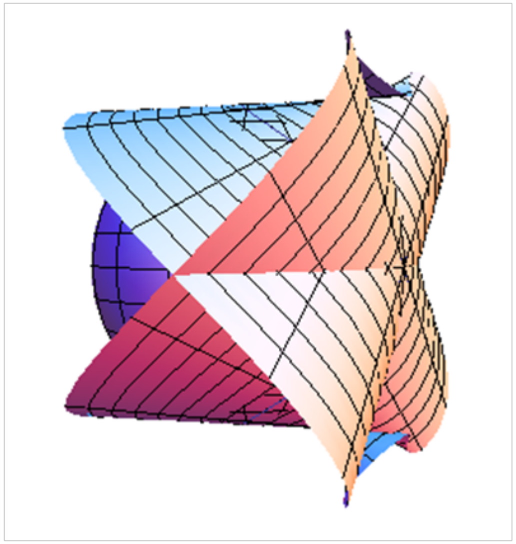
$$\mathbf{x}(u,v) = \beta(u) + v\delta(u)$$

$$= (-\frac{1}{2}(-2\cos u + v(3+\cos 2u))\sin u,$$

$$v\cos^3 u + \sin^2 u, \cos u + v\sin^2 u)$$

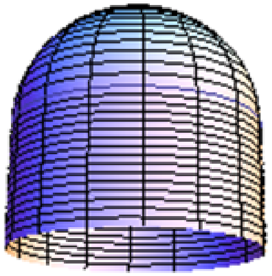






Further Research

- Description of TED surfaces as the union a region *R* of the sphere and the developable ruled surface corresponding to the boundary of *R*



- Must every (smooth) TED surface with constant pedal be either the sphere, a tangent plane, a developable ruled surface, or a union of a region of the sphere and a developable rule surface?

- Generalization to *n*-dimensional TED manifolds

References

A. Fabian, English Translation of Leonard Euler's E236 publication, *Exposition de quelques paradoxes dans le calcul integral* (Explanation of Certain Paradoxes in Integral Calculus), originally published in *Memoires de l'academie des sciences de Berlin* 12, 1758, pp. 300-321; also published in *Opera Omnia*: Series 1, Volume 22, pp. 214 – 236. Posted on the Euler Archive: http://www.math.dartmouth.edu/~euler/

A. Fabian and H. D. Nguyen, *Paradoxical Euler: Integrating by Differentiating*, Preprint (2008).

E. Sandifer, *Curves and paradox*, How Euler Did It (October 2008 column article), MAA Online: http://www.maa.org/editorial/euler/HEDI%2060%20Curves%20and%20paradox.pdf