Discovering **Bernoulli Number Identities** *via* **Euler-Maclaurin Summation**

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Bernoulli Polynomials

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

...

The Bernoulli *numbers* are defined by $B_n = B_n(0)$:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \; \ldots$$

Generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Recursive formula:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n$$

Euler-Maclaurin Summation (EMS)

Appell sequence:

(i) $B_0(x) = 1$ (ii) $B_n'(x) = nB_{n-1}(x)$

(iii) $\int_0^1 B_n(x)dx = \delta_{n,0} = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

EMS:

$$\sum_{k=1}^{n-1} f(k) = \int_0^n f(x)dx - \frac{1}{2}[f(n) + f(0)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right]$$

Special Case: Assume n = 1 and f a polynomial of degree p

$$\boxed{\int_0^1 f(x)dx = \frac{1}{2}[f(1) + f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]}$$

Hunt for Bernoulli Number Identities

Example 1: $f(x) = x^{m-1} \quad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 x^{m-1}dx = \frac{1}{m} \qquad \frac{1}{2}[f(1)+f(0)] = \frac{1}{2}[1+0] = \frac{1}{2}$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = \frac{(m-1)!}{(m-k)!}\delta_{mk}$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1)+f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$\frac{1}{m} = \frac{1}{2} - \sum_{k=2}^{m-1} \frac{B_k}{k!}\left[\frac{(m-1)!}{(m-k)!}\right] \Rightarrow 1 - \frac{m}{2} + \sum_{k=2}^{m-1} \frac{m!}{k!(m-k)!} B_k = 0$$

$$\boxed{\sum_{k=0}^{m-1} \binom{m}{k} B_k = 0} \qquad \text{(Compare with } \sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n \text{)}$$

Example 2: $f(x) = B_m(x) \quad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 B_m(x)dx = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m > 0 \end{cases} \qquad \frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[B_m(1) + B_m(0)]$$

$$= B_m$$

Derivatives of Bernoulli polynomials

$$B_m'(x) = mB_{m-1}(x) = \frac{m!}{(m-1)!}B_{m-1}(x) \Rightarrow B_m^{(k)}(x) = \frac{m!}{(m-k)!}B_{m-k}(x)$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = B_m^{(k-1)}(1) - B_m^{(k-1)}(0) = m!\delta_{mk}$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1) + f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$0 = B_m - B_m \quad \text{(Trivial!)}$$

Example 3: $f(x) = (1-x)B_m(x) \qquad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 (1-x)B_m(x)dx = -\frac{B_{m+1}}{m+1}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[0 + B_m(0)] = \frac{1}{2}B_m$$

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}[1-x] \cdot B_m^{(n-k)}(x)$$

$$= (1-x)\frac{m!}{(m-n)!}B_{m-n}(x) - n\frac{m!}{(m-n+1)!}B_{m-n+1}(x)$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = -\frac{m!}{(m-k+1)!}B_{m-k+1} + 2(k-1)\frac{m!}{(m-k+2)!}B_{m-k+2}\delta_{m,k-1}$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1)+f(0)] - \sum_{k=2}^{p} \frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$-\frac{B_{m+1}}{m+1} = \frac{1}{2}B_m - \frac{2mB_1B_{m+1}}{m+1} + \frac{1}{(m+1)}\sum_{k=2}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$-mB_{m+1} = (m+1)B_m + \sum_{k=0}^{m+1}\binom{m+1}{k}B_kB_{m-k+1}$$

$$\boxed{\sum_{k=0}^{m+1}\binom{m+1}{k}B_kB_{m-k+1} = -(m+1)B_m - mB_{m+1}}$$

Euler’s formula (assume $m+1=2n>2$):

$$\boxed{\sum_{k=0}^{n}\binom{2n}{2k}B_{2k}B_{2n-2k} = -(2n-1)B_{2n}}$$

Hypergeometric Bernoulli Polynomials (N = 2)

$$B_0(2,x) = 1, \qquad B_1(2,x) = x - \frac{1}{3},$$

$$B_2(2,x) = x^2 - \frac{2}{3}x + \frac{1}{18}, \quad B_3(2,x) = x^3 - x^2 + \frac{1}{6}x + \frac{1}{90},$$

...

Define *hypergeometric Bernoulli numbers* by $B_n(2) = B_n(2,0)$

$$B_0(2) = 1, \quad B_1(2) = -\frac{1}{3}, \quad B_2(2) = \frac{1}{18}, \quad B_3(2) = \frac{1}{90}$$

Appell sequence with zero first moment:

(i) $B_0(2,x) = 1$

(ii) $B_n\,'(2,x) = nB_{n-1}(2,x)$

(iii) $\int_0^1 (1-x)B_n(2,x)dx = \begin{cases} 1/2 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$

Example 5: $f(x) = B_m(2, x) \qquad (m > 1)$

$$\int_0^1 f(x)dx = \int_0^1 B_m(2, x)dx = B_m(2) + \frac{m}{2}\delta_{m,1}$$

$$\frac{1}{2}[f(1) + f(0)] = \frac{1}{2}[B_m(2,1) + B_m(2,0)] = B_m(2) + \frac{m}{2}B_{m-1}(2) \quad (m > 1)$$

$$f^{(k-1)}(1) - f^{(k-1)}(0) = \frac{m!}{(m-k)!}B_{m-k}(2) + \frac{m!}{2(m-k-1)!}\delta_{m-1,k}$$

EMS:

$$\int_0^1 f(x)dx = \frac{1}{2}[f(1) + f(0)] - \sum_{k=2}^{p}\frac{B_k}{k!}[f^{(k-1)}(1) - f^{(k-1)}(0)]$$

$$B_m(2) = B_m(2) + \frac{m}{2}B_{m-1}(2) - \frac{mB_{m-1}}{2} - \sum_{k=2}^{m}\binom{m}{k}B_k B_{m-k}(2)$$

Theorem:

$$\sum_{k=0}^{m}\binom{m}{k}B_k B_{m-k}(2) = -\frac{mB_{m-1}}{2} + B_m(2)$$

Efficient formula for $B_m(2)$:

$$B_m(2) = B_m + \frac{2}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k}B_k(2)B_{m+1-k}$$

$$= \begin{cases} B_m + \dfrac{2}{m+1}\displaystyle\sum_{j=0}^{\frac{m-2}{2}}\binom{m+1}{2j+1}B_{2j+1}(2)B_{m-2j} & \text{if } m \text{ even} \\ B_m + \dfrac{2}{m+1}\displaystyle\sum_{j=0}^{\frac{m-1}{2}}\binom{m+1}{2j}B_{2j}(2)B_{m+1-2j} & \text{if } m \text{ odd} \end{cases}$$

Interesting Problem: Find a formula for $B_m(2)$ in terms of only Bernoulli numbers.

Further Exploration

Other choices for $f(x)$:

$$f(x) = (1-x)B_m(x)B_n(x) \quad \text{(Cubic recurrence formula?)}$$

$$f(x) = E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] \quad \text{(Euler polynomials)}$$

$$f(x) = H_n(x) = (-1)^n e^{x^2/2}\frac{d^n(e^{-x^2/2})}{dx^n} \quad \text{(Hermite polynomials)}$$

References

[1] L. Euler, [E212] Institutiones calculi differentialis (1713), Part II, Chapter 5, Translation by David Pengelley, available at The Euler Archive: http://www.math.dartmouth.edu/~euler/

[2] A. Hassen and H. Nguyen, *Hypergeometric Bernoulli Polynomials and Appell Sequences*, to appear in Intern. J. Number Theory.