Hypergeometric Zeta Functions

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Joint Mathematics Meeting
Atlanta, GA
1-5-05

Euler-Riemann Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}(s) > 1$$

Theorem (Riemann, 1859)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \qquad \mathrm{Re}(s) > 1$$

Gamma Function (Euler)

$$\Gamma(s) \equiv \int_0^{\infty} x^{s-1} e^{-x} dx = \int_0^{\infty} \frac{x^{s-1}}{e^x} dx$$

Hypergeometric Zeta Function

$$\zeta_2(s) \equiv \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{e^x - 1 - x} dx$$

Lemma

(a) $\zeta_2(s)$ converges for $\mathrm{Re}(s) > 1$

(b) $\zeta_2(s) = \sum_{n=1}^{\infty} U(s+1, s+1+n, n)$ (Hypergeometric Zeta Function)

Confluent Hypergeometric Function of the Second Kind:

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^\infty x^{a-1}(1+x)^{b-a-1} e^{-zx} dx$$

Proof of (b):

1. Express integrand as a binomial series:

$$\frac{1}{e^x - 1 - x} = \frac{e^{-x}}{1 - e^{-x}(1+x)} = e^{-x}\sum_{n=0}^{\infty}[e^{-x}(1+x)]^n = \sum_{n=1}^{\infty} e^{-nx}(1+x)^{n-1}$$

2. Reverse order of integration and summation (uniform convergence):

$$\zeta_2(s) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{e^x - 1 - x}dx$$

$$= \frac{1}{\Gamma(s+1)}\int_0^\infty x^s \sum_{n=1}^{\infty} e^{-nx}(1+x)^{n-1}dx$$

$$= \sum_{n=1}^{\infty}\frac{1}{\Gamma(s+1)}\int_0^\infty x^s(1+x)^{n-1}e^{-nx}dx \qquad (*)$$

$$= \sum_{n=1}^{\infty} U(s+1, s+1+n, n)$$

Theorem

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{\mu(s,n)}{n^{s+1}}$$

where

$$\mu(s,n) \equiv \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(s+1)_k}{n^k}$$

Proof:

1. Express integrand in (*) as a binomial series:

$$x^s (1+x)^{n-1} e^{-nx} = x^s \left[\sum_{k=0}^{n-1} \binom{n-1}{k} x^k \right] e^{-nx}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{s+k} e^{-nx}$$

2. Evaluate integral using the substitution $nx \to x$

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{1}{\Gamma(s+1)} \int_0^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{s+k} e^{-nx} dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{\Gamma(s+1)} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{n^{s+k+1}} \int_0^{\infty} x^{s+k} e^{-x} dx$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{n^{s+k+1}} \frac{\Gamma(s+k+1)}{\Gamma(s+1)}$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(s+1)(s+2)\cdots(s+k)}{n^{s+k+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{s+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(s+1)_k}{n^k}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(s,n)}{n^{s+1}}$$

Lemma

$$\mu(1,n) = n$$

Or equivalently,

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(2)_k}{n^k} = n$$

Expression for n = 4:

$$1 + (4-1)\frac{2}{4} + \frac{(4-1)(4-2)}{2}\frac{2\cdot 3}{4^2} + \frac{(4-1)(4-2)(4-3)}{2\cdot 3}\frac{2\cdot 3\cdot 4}{4^3} = 4$$

Corollary

$$\zeta_2(s) > \zeta(s) \qquad \mathrm{Re}(s) > 1$$

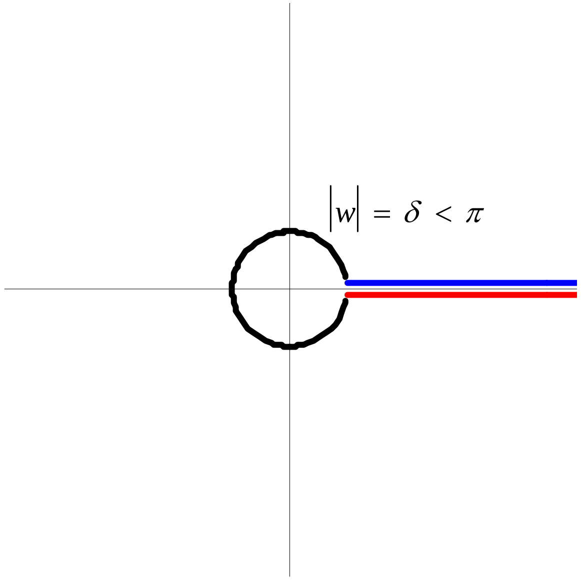
Proof:

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{\mu(s,n)}{n^{s+1}} > \sum_{n=1}^{\infty} \frac{\mu(1,n)}{n^{s+1}} = \sum_{n=1}^{\infty} \frac{n}{n^{s+1}} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

Analytic Continuation

Contour Integration

$$I(z) \equiv \int_\gamma (e^w - 1)^{-1} (-w)^z \frac{dw}{w}$$



Theorem (Riemann)

(a) $I(z)$ is analytic for all z

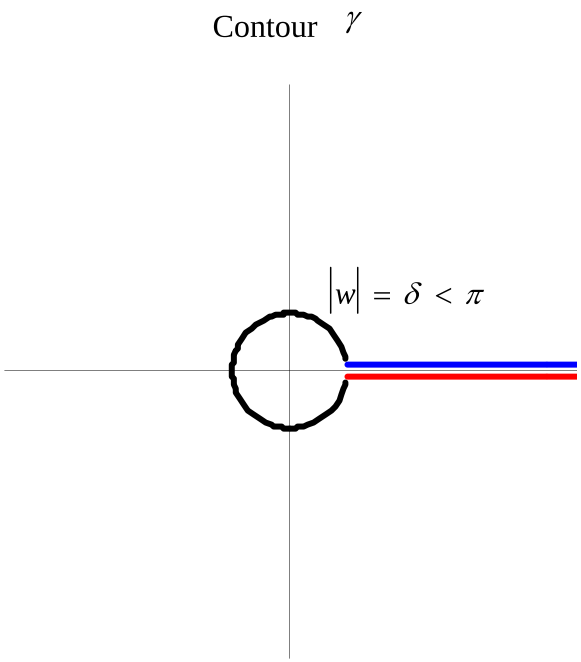
(b) $I(k) = 0 \qquad (k = 2, 3, 4, \ldots)$

(c) $\zeta(z) = \dfrac{\Gamma(1-z)}{2\pi i} I(z)$

Note: Part (c) gives the analytic continuation of zeta to all $z \neq 1$

Hypergeometric version:

$$I_2(z) \equiv \int_\gamma (e^w - 1 - x)^{-1} (-w)^{z+1} \frac{dw}{w}$$



Theorem

(a) $I_2(z)$ is analytic for all z

(b) $I_2(k) = 0 \qquad (k = 2, 3, 4, \ldots)$

(c) $\zeta_2(z) = \dfrac{\Gamma(-z)}{2\pi i} I_2(z)$

Note: Part (c) gives the analytic continuation of zeta to all $z \neq 0,1$

Bernoulli Numbers and Trivial Zeros of Zeta

Theorem (Riemann)

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \qquad (n = 1, 2, 3, \ldots)$$

Proof:

(i) $$\zeta(-n) = \frac{\Gamma(1+n)}{2\pi i} I(-n) = \frac{n!}{2\pi i} I(-n)$$

(ii) $$I(-n) = \oint_{|w|=\delta} \left(\frac{w}{e^w - 1} \right) (-w)^{-n-2} \, dw$$

$$= (-1)^n \oint_{|w|=\delta} \left(\sum_{m=0}^{\infty} \frac{B_m}{m!} w^m \right) \frac{dw}{w^{n+2}}$$

$$= (-1)^n \frac{2\pi i}{(n+1)!} B_{n+1}$$

Corollary (Trivial Zeros of Zeta):

$$\zeta(-2n) = 0$$

Bernoulli Numbers:

$$\frac{w}{e^w - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} w^m$$

$$= 1 - \frac{1}{2} w + \frac{1}{12} w^2 - \frac{1}{72} w^4 + \frac{1}{30240} w^6 - \ldots$$

$B_0 = 1$ $\quad$ $B_1 = -1/2$

$B_2 = 1/6$ $\quad$ $B_3 = 0$

$B_4 = -1/30$ $\quad$ $B_5 = 0$

$B_6 = 1/42$ $\quad$ $B_7 = 0$

Hypergeometric version

Theorem

$$\zeta_2(-n) = (-1)^{n-1} \frac{2}{n(n+1)} A_{n+1} \qquad (n = 1, 2, 3, \ldots)$$

Note: No trivial zeros in this case!

Howard (Generalized Bernoulli) Numbers:

$$\frac{w^2/2}{e^w - 1 - x} = \sum_{m=0}^{\infty} \frac{A_m}{m!} w^m$$

$$= 1 - \frac{1}{3} w + \frac{1}{36} w^2 + \frac{1}{540} w^3 - \frac{1}{6480} w^4 - \frac{1}{27216} w^5 + \ldots$$

$A_0 = 1$ $\qquad$ $A_1 = -1/3$

$A_2 = 1/18$ $\qquad$ $A_3 = 1/90$

$A_4 = -1/270$ $\qquad$ $A_3 = -5/1134$

Functional Equation

Lemma (Riemann)

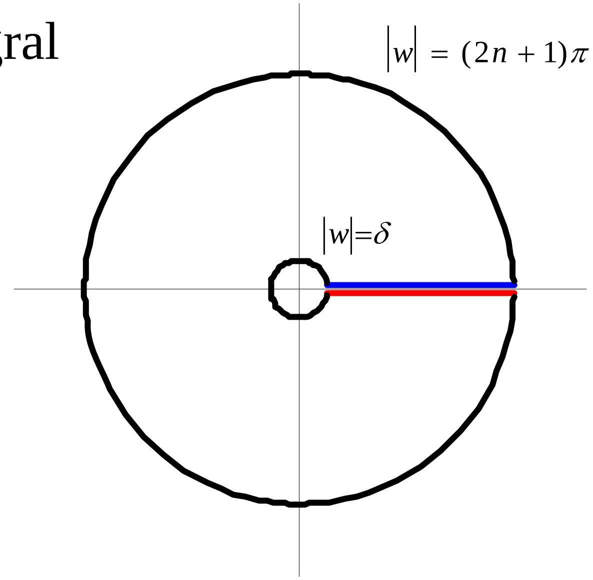
$$\frac{I(z)}{2\pi i} = 2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\zeta(1-z)$$

Proof:

1. Consider the 'truncated' contour integral

$$I(n,z) = \int_{\gamma_n} (e^w - 1)^{-1}\frac{(-w)^z}{w}\,dw$$

Contour γ_n

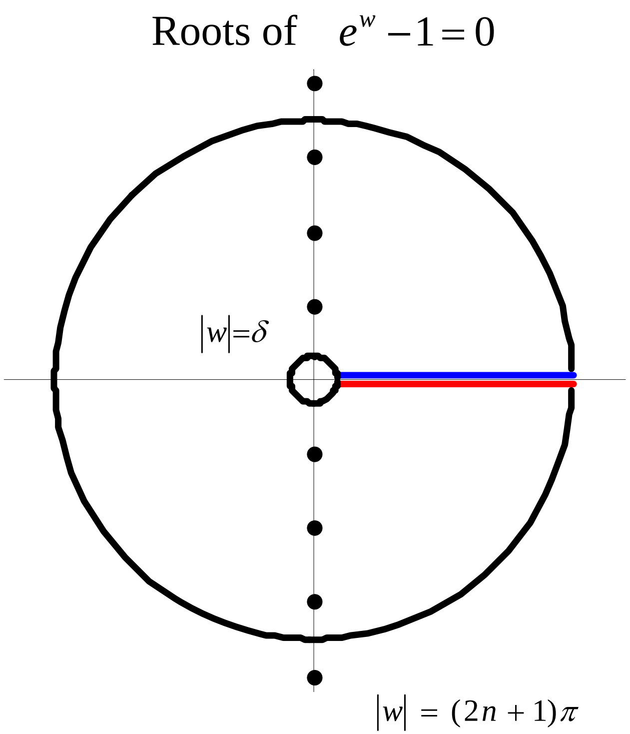


2. Evaluate the integral using residues:

$$\frac{I(n,z)}{2\pi i} = \sum_{\substack{k=-n \\ k\neq 0}}^{n} \operatorname{Res}\left((e^w - 1)^{-1} \frac{(-w)^z}{w}, w = 2\pi i k \right)$$

$$= -\sum_{k=1}^{n} \left[(2\pi i k)^{z-1} + (-2\pi i k)^{z-1} \right]$$

$$= 2(2\pi)^{z-1} \sin\left(\frac{\pi z}{2}\right) \sum_{k=1}^{n} k^{z-1}$$



3. Let $n \to \infty$

$$\frac{I(z)}{2\pi i} = \lim_{n\to\infty} \frac{I(n,z)}{2\pi i}$$

$$= 2(2\pi)^{z-1} \sin\left(\frac{\pi z}{2}\right) \zeta(1-z)$$

Theorem (Euler-Riemann Functional Equation)

$$\zeta(z) = 2(2\pi)^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)$$

Corollary (Euler)

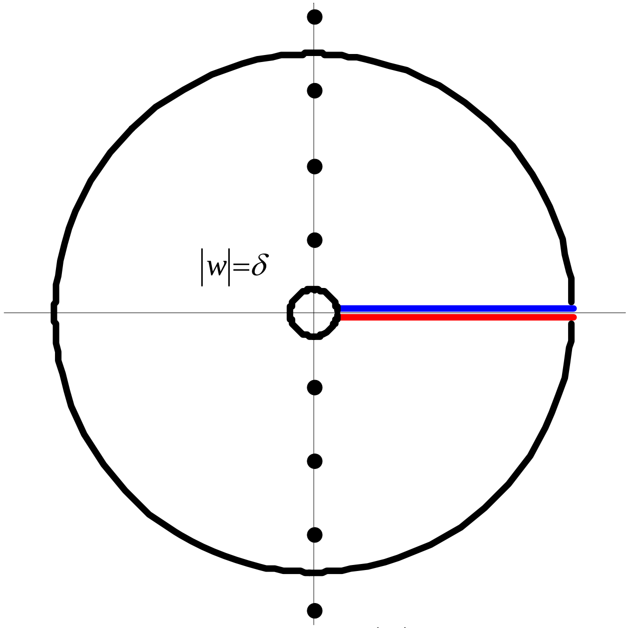
$$\zeta(2n) = \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

Hypergeometric version:

Lemma

$$\frac{I_2(z)}{2\pi i} = 2(-1)^{z+1}\sum_{k=1}^{\infty} r_k^{z-1}\cos[(z-1)\theta_k]$$

where $w_k = r_k e^{i\theta_k}$ are the roots of $e^w - 1 - x = 0$



Proof:

1. Consider the 'truncated' contour integral

$$I_2(n,z) \equiv \int_{\gamma_n} (e^w - 1 - x)^{-1} \frac{(-w)^{z+1}}{w} dw$$

2. Evaluate the integral using residues:

$$\frac{I_2(n,z)}{2\pi i} = \sum_{\substack{k=-n \\ k\neq 0}}^{n} \mathrm{Res}\left((e^w - 1 - x)^{-1} \frac{(-w)^{z+1}}{w}, w = w_k \right)$$

$$= (-1)^{z+1} \sum_{k=1}^{n} \left[(w_k)^{z-1} + (\overline{w}_k)^{z-1} \right]$$

$$= 2(-1)^{z+1} \sum_{k=1}^{n} r_k^{z-1} \cos[(z-1)\theta_k], \qquad w_k = r_k e^{i\theta_k}$$

3. Let $n \to \infty$

$$\frac{I_2(z)}{2\pi i} = \lim_{n\to\infty} \frac{I_2(n,z)}{2\pi i}$$

$$= 2(-1)^{z+1} \sum_{k=1}^{\infty} r_k^{z-1} \cos[(z-1)\theta_k]$$

Theorem For $\mathrm{Re}(z) < 0,$

$$\zeta_2(z) = 2(-1)^{z+1}\Gamma(-z)\sum_{k=1}^{\infty} r_k^{z-1}\cos[(z-1)\theta_k]$$

Corollary For $\mathrm{Re}(z) < 0,$

$$\left|\zeta_2(z)\right| < 2(2\pi)^{\mathrm{Re}(z)-1} e^{\mathrm{Im}(s)\cdot\pi/2} \left|\Gamma(-z)\right| \zeta(1-\mathrm{Re}(z))$$

Complex roots of $e^w - 1 - w = 0:$

$$w_n = a_n + b_n i w_k = r_k e^{i\theta_k} \qquad w_1 \approx 2.09 + 7.46i$$

Lemma (Howard)

$$(2n+1/4)\pi < b_n < (2n+1/2)\pi$$

Generalization to Higher-Orders

(Joint work with Abdul Hassen)

N^{th}-Order Hypergeometric Zeta Functions

$$\zeta_N(s) \equiv \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\,dx$$

where

$$T_N(x) = \sum_{k=0}^{N} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}$$

Note: Many of the results discussed earlier generalize to N^{th}-order hypergeometric zeta functions.

Open Problems

1. Do hypergeometric zeta functions satisfy a functional equation? The answer will require understanding the roots of $e^w - 1 - w = 0.$

2. Do hypergeometric zeta functions have nontrivial zeros? If so, locate them.

3. Find efficient numerical algorithms for evaluating hypergeometric zeta functions. Poor estimate for $\zeta_2(2)$:

$$2.33 < \zeta_2(2) = \sum_{n=1}^{\infty} \frac{\mu(2,n)}{n^2} < 2.51$$

Compare with $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449$

References

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