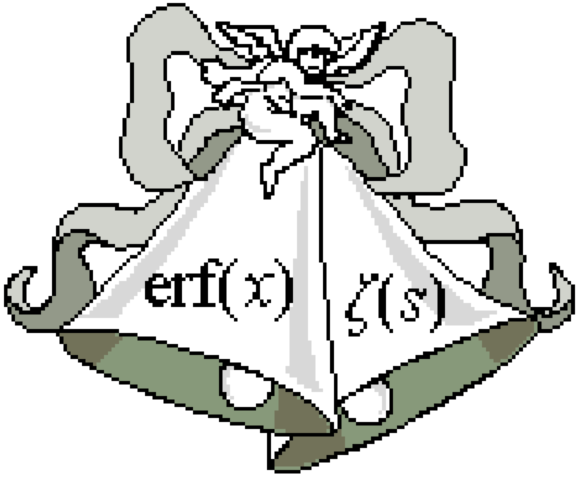
# A Half-Marriage in Error:
## When Zeta Integrates a Gaussian

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## Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

## Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

## Error Zeta Function

$$\zeta_{1/2}(s) = \frac{2}{\Gamma(s-1/2)} \int_0^{\infty} \frac{x^{2(s-1)} e^{-x^2}}{\operatorname{erf}(x)} dx \qquad (s = \sigma + it)$$

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$$\Gamma(s) \equiv \int_0^{\infty} x^{s-1} e^{-x} dx = \int_0^{\infty} \frac{x^{s-1}}{e^x} dx$$

# Zeta Clan

Hypergeometric Zeta Functions:

$$\zeta_N(s) \equiv \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx \qquad N \in \mathbb{N}$$

where

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}$$

Examples:

$$N = 1: \qquad \zeta_1(s) = \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \qquad \text{(Classical zeta)}$$

$$N = 2: \qquad \zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^s}{e^x - 1 - x} dx$$

Connection with Hypergeometric Series:

$$e^x - T_{N-1}(x) = \frac{x^{N-1}}{\Gamma(N)} \left[ {}_1F_1(1, N, x) - 1 \right], \quad N \in \mathbb{N}$$

# Fractional Descendants

Hypergeometric Series ($a$, $b > 0$):

$$ {}_1F_1(a,b;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{x^n}{n!} = 1 + \frac{a}{b} \cdot x + \frac{a(a+1)}{b(b+1)} \cdot \frac{x^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \cdot \frac{x^2}{2!} + \ldots $$

Fractional Hypergeometric Zeta:

$$ \zeta_N(s) \equiv \frac{\Gamma(N)}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s-1}}{{}_1F_1(1,N;x)-1}\, dx \qquad N \in (0,\infty) $$

Error Zeta ($N$ = 1/2):

$$ \zeta_{1/2}(s) = \frac{\Gamma(1/2)}{\Gamma(s-1/2)} \int_0^\infty \frac{x^{s-1}}{{}_1F_1(1,1/2;x)-1}\, dx $$

$$ = \frac{1}{\Gamma(s-1/2)} \int_0^\infty \frac{x^{s-3/2} e^{-x}}{\operatorname{erf}(\sqrt{x})}\, dx $$

$$ = \frac{2}{\Gamma(s-1/2)} \int_0^\infty \frac{x^{2(s-1)} e^{-x^2}}{\operatorname{erf}(x)}\, dx $$

# Family Traits

I. Classical Zeta:

$$\lim_{s\to 1}\left[\zeta(s)-\frac{1}{s-1}\right]=-\frac{\Gamma'(1)}{\Gamma(1)}=\gamma\approx 0.577$$

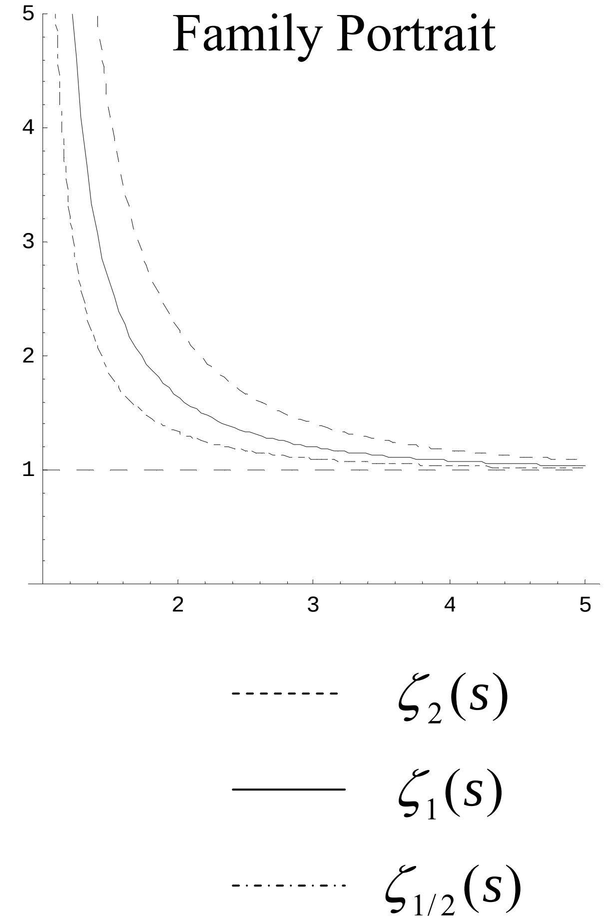
II. Hypergeometric Zeta:

$$\lim_{s\to 1}\left[\zeta_N(s)-\frac{N!}{s-1}\right]=\log(N!)-N\frac{\Gamma'(N)}{\Gamma(N)}$$

III. Error Zeta:

$$\lim_{s\to 1}\left[\zeta_{1/2}(s)-\frac{1/2}{s-1}\right]=\log\frac{\sqrt{\pi}}{2}-\frac{\Gamma'(1/2)}{2\Gamma(1/2)}$$

Family Portrait

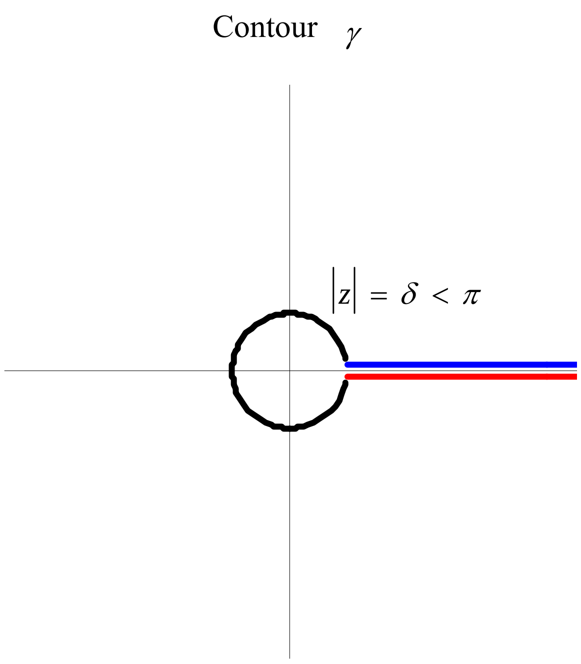


# A Vast Domain

Analytic Continuation to Re($s$) < 1

I. Classical Zeta:

$$I(s) \equiv \int_\gamma \frac{(-z)^s}{(e^z - 1)} \frac{dz}{z}$$




**Theorem** (Riemann)

(a) $I(s)$ is analytic for all $s$

(b) $I(k) = 0 \qquad (k = 2, 3, 4, \ldots)$

(c) $\zeta(s) = \Gamma(1-s)\dfrac{I(s)}{2\pi i}$

(d) $\zeta(s)$ has one simple pole at $s = 1$

## II. Hypergeometric Zeta:

$$I_N(s) \equiv \int_\gamma \frac{(-z)^{s+N-1}}{(e^z - T_{N-1}(z))} \frac{dz}{z}$$

**Theorem**

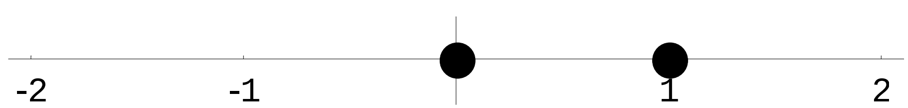
(a) $I_N(s)$ is analytic for all $s$

(b) $I_N(k) = 0 \qquad (k = 2, 3, 4, \ldots)$

(c) $\zeta_N(s) = \Gamma(1-(s+N-1)) \dfrac{I_N(s)}{2\pi i}$

(d) $\zeta_N(s)$ has $N$ simple poles at $s = 1, 0, -1, \ldots, 2-N$

$(N = 2)$



# III. Error Zeta:

$$I_{1/2}(s) \equiv \int_{\gamma} \frac{(-z)^{2s-1} e^{-z^2}}{\mathrm{erf}(z)} \frac{dz}{z}$$

**Theorem**

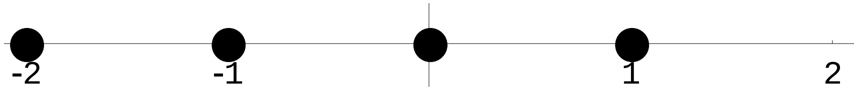
(a) $I_{1/2}(s)$ is analytic for all $s$

(b) $I_{1/2}(n/2) = 0 \qquad (n = 3, 4, \ldots)$

(c) $\zeta_{1/2}(s) = \dfrac{\Gamma(1-(s-1/2))}{2\cos[\pi(s-1/2)]} I_{1/2}(s)$

(d) $\zeta_{1/2}(s)$ has infinitely many simple poles at $s = 1, 0, -1, \ldots$

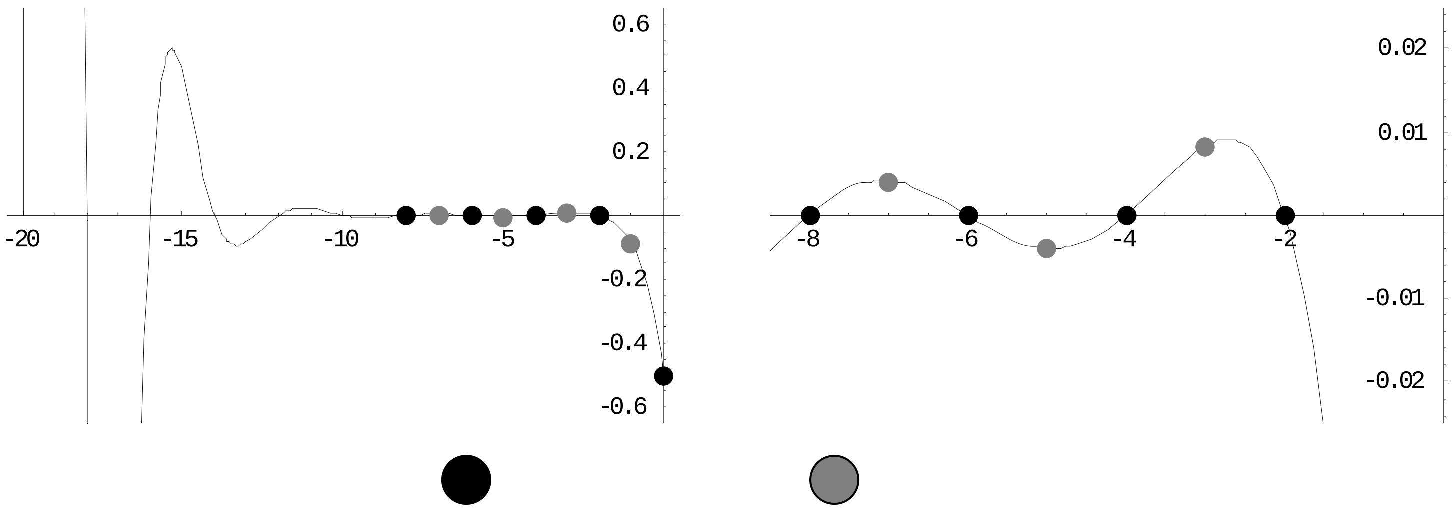
$(N = 1/2)$



# Property Values

## Zeta Values at Negative Integers

I. Classical Zeta ($s < 0$):




$$\zeta(0) = -1/2 \qquad \zeta(-1) = -1/12$$

$$\zeta(-2) = 0 \qquad \zeta(-3) = 1/120$$

$$\zeta(-4) = 0 \qquad \zeta(-5) = -1/252$$

$$\zeta(-6) = 0 \qquad \zeta(-7) = 1/240$$

**Theorem** (Euler-Riemann)

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \qquad (n = 0, 1, 2, 3, \ldots)$$

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Bernoulli Numbers:

$$\frac{z}{e^z - 1} = \sum_{m=0}^{\infty} B_m \frac{z^m}{m!} = 1 - \frac{1}{2} z + \frac{1}{6} \cdot \frac{z^2}{2!} + 0 \cdot \frac{z^3}{3!} - \frac{1}{30} \cdot \frac{z^4}{4!} + 0 \cdot \frac{z^5}{5!} + \frac{1}{42} \cdot \frac{z^6}{6!} - \ldots$$

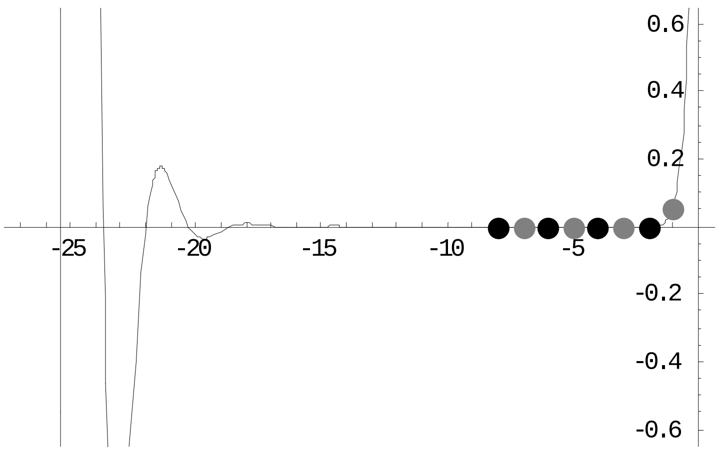
| | |
|---|---|
| $B_1 = -1/2$ | $B_2 = 1/6$ |
| $B_3 = 0$ | $B_4 = -1/30$ |
| $B_5 = 0$ | $B_6 = 1/42$ |
| $B_7 = 0$ | $B_8 = -1/30$ |

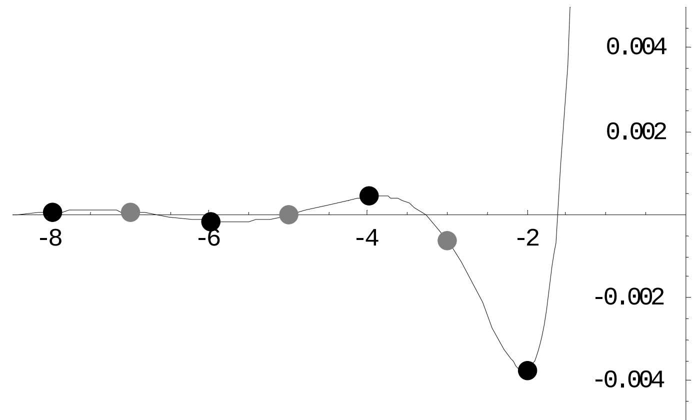
**Corollary** (Trivial Zeros of Classical Zeta):

$$\zeta(-2n) = 0$$

# II. Hypergeometric Zeta ($s < 0$):

$N = 2$:




| ● | ● (grey) |
|---|---|
| $\zeta_2(0) = \text{Undefined}$ | $\zeta_2(-1) = 1/18$ |
| $\zeta_2(-2) = -1/270$ | $\zeta_2(-3) = -1/1620$ |
| $\zeta_2(-4) = 1/2268$ | $\zeta_2(-5) = -1/85050$ |
| $\zeta_2(-6) = -1/7290$ | $\zeta_2(-7) = 13/204120$ |

**Theorem** ($N = 2$)

$$\zeta_2(-n) = \frac{2(-1)^{n-N+1}}{n(n+1)} B_{N,n+1}(2) \qquad (n = 1, 2, 3, \ldots)$$

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Generalized Bernoulli Numbers:

$$\frac{z^N / N!}{e^w - T_{N-1}(x)} = \sum_{m=0}^{\infty} B_{N,m} \frac{z^m}{m!}$$

$$= 1 - \frac{1}{3} z + \frac{1}{18} \cdot \frac{z^2}{2!} + \mathbf{\frac{1}{90}} \cdot \frac{z^3}{3!} - \frac{1}{270} \cdot \frac{z^4}{4!} - \mathbf{\frac{5}{1134}} \cdot \frac{z^5}{5!} + \ldots$$

$B_m(N)$ - generalized Bernoulli numbers

**Corollary** ("Trivial" Zeros): $\zeta_2(s) \neq 0$ on the left half-plane $\{\mathrm{Re}(s) < \sigma_2 \approx -2.4\}$, except for infinitely many zeros located on the negative real axis.

## III. Error Zeta ($s < 0$):

**Theorem** ($N = 1/2$): $\zeta_{1/2}(s)$ has simples poles at negative integers (including 0 and 1) and has zeros at negative half-integers (including 1/2). In particular:

(a) $$\mathrm{Res}(\zeta_{1/2}(s), s = n) = \frac{(-1)^{2n-1} B_{1/2,2-2n}}{2(2-2n)!\Gamma(n-1/2)} \qquad (n = 1, 0, -1, \ldots)$$

(b) $$\zeta_{1/2}((2n+1)/2) = 0 \qquad (n = 0, -1, -2, \ldots)$$

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Fractional Bernoulli Numbers:

$$\frac{ze^{-z^2}}{\mathrm{erf}(z)} = \sum_{m=0}^{\infty} B_{1/2,m} \frac{z^m}{m!} = \frac{\sqrt{\pi}}{2}\left(1 - \frac{\mathbf{4}}{\mathbf{3}} z^2 + \frac{\mathbf{64}}{\mathbf{15}} \cdot \frac{z^4}{4!} - \frac{\mathbf{256}}{\mathbf{21}} \cdot \frac{z^6}{6!} + \ldots\right)$$

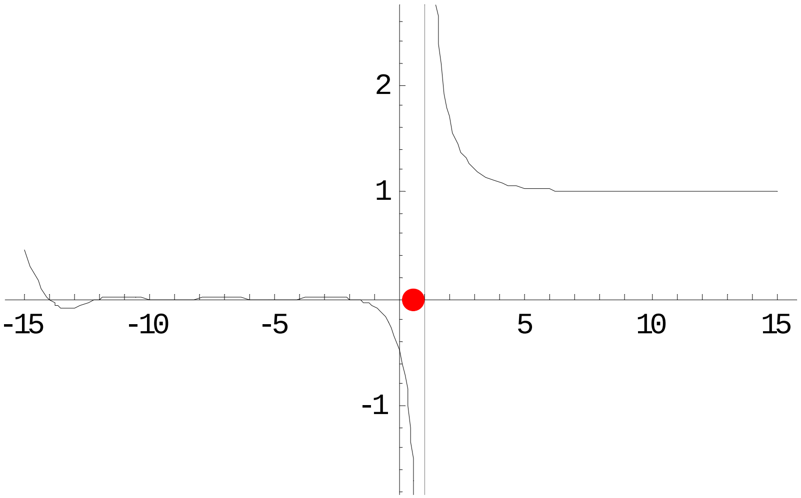
$B_{1/2,m}$ - fractional Bernoulli numbers

# Best of All Possible Worlds?

A Functional Equation

I. Classical Zeta:

Reflection across the critical line Re($s$) = 1/2:




**Theorem** (Euler-Riemann Functional Equation)

$$\zeta(s) = 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

**Corollary** (Euler)

$$\zeta(2n) = \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}$$

# II. Hypergeometric Zeta:

**Theorem** For $\mathrm{Re}(s) < 0$,

$$\zeta_N(s) = 2(-1)^{N-1}(N-1)!\Gamma(1-(s+N-1))\sum_{k=1}^{\infty} r_k^{s-1}\cos[(s-1)(\theta_k-\pi)]$$

**Functional Inequality Theorem** $(N=2)$ For $\mathrm{Re}(s) < 0$,

$$|\zeta_2(s)| < 2(2\pi)^{\mathrm{Re}(s)-1} e^{|\mathrm{Im}(s)(\theta_1-\pi)|} |\Gamma(-s)| \zeta_2(1-\mathrm{Re}(s))$$

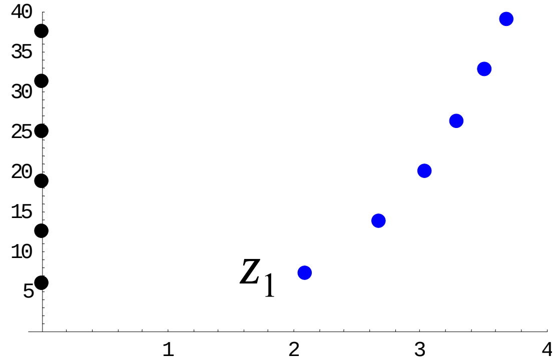
---

$z_n = a_n + b_n i = r_k e^{i\theta_k}$ are complex roots of

$$e^z - 1 - z = 0$$

$z_1 \approx 2.0888 + 7.4615i,$

$r_1 \approx 7.7484, \ \ \theta_1 \approx 74.3604^\circ$



III. Error Zeta:

**Theorem** For Re($s$) $< 0$,

$$\zeta_{1/2}(s)=\frac{\sqrt{\pi}\,\Gamma(1-(s-1/2))}{2\cos[\pi(s-1/2)]}\sum_{k=1}^{\infty} r_k^{2s-2}\{\cos[2(s-1)(\pi-\theta_k)]+\cos[2(s-1)\theta_k]\}$$

**Functional Inequality Theorem** ($N$ = 1/2) For Re($s$) $< 0$,

$$\left|\zeta_{1/2}(s)\right| < (2\pi)^{\mathrm{Re}(s)-1/2}\left|\frac{\Gamma(1-(s-1/2))}{\sqrt{2}\cos[\pi(s-1/2)]}\right|\zeta(1-\mathrm{Re}(s),3/4)$$

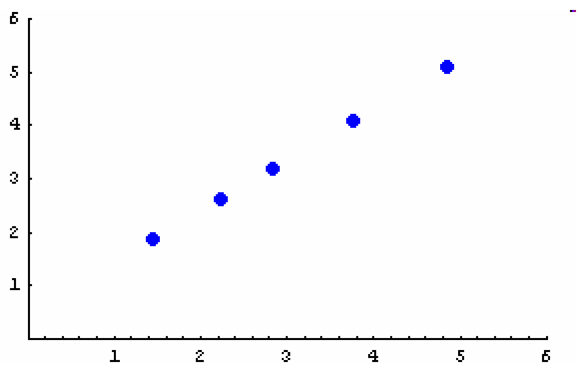
---

$z_n = a_n + b_n i = r_k e^{i\theta_k}$ are complex roots of

$$\mathrm{erf}(z)=0$$

$z_1 \approx 1.4506 + 1.8809i$,

$r_1 \approx 2.3753,\ \ \theta_1 \approx 52.3596^\circ$




Hurwitz Zeta Function: $\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$

# What Does the Future Hold?

## Open Problems

1. Do fractional zeta functions satisfy a functional *equation* ?
2. Locate nontrivial zeros of fractional zeta functions.

## References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