Newton's "Greatest Blunder":

His theorem on revolving orbits and failed calculation of lunar precession

A clever man commits no minor blunders.

--- Johann Wolfgang Von Goethe (1749 - 1832)

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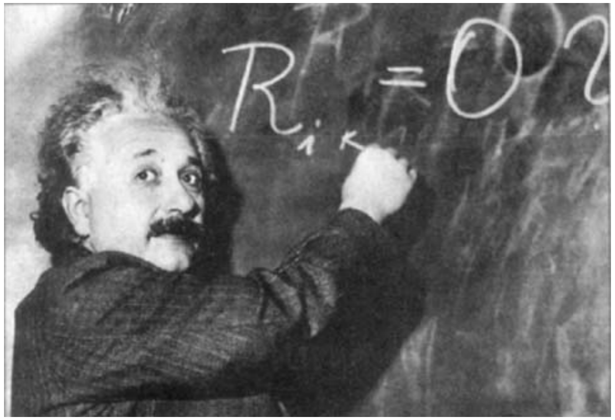
Einstein’s Greatest Blunder (1917)

…Thus, Einstein’s original gravity equation was correct, and changing it was a mistake. Much later, when I was discussing cosmological problems with Einstein, he remarked that the introduction of the *cosmological constant term was the biggest blunder he ever made in his life*. But this “blunder”, rejected by Einstein, is still sometimes used by cosmologists even today, and the *cosmological constant* Λ *rears its ugly head again and again and again*.

--- George Gamow (1904-1968), *My World Line*, 1970, p. 44.



Albert Einstein (1879-1955)
http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Einstein.html

General Relativity (1915)

Einstein's field equations (16 coupled PDE's)

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}$$

(Curvature of space = Gravitation field of matter)

- Perihelion advance of planetary orbits (Mercury)
- Deflection of light (Eddington, 1919)
- Expanding universe (Hubble, 1929)

Cosmological constant Λ (expansion rate):

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

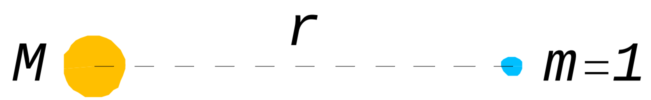
- Quantum vacuum energy
- Dark matter

Objectives of This Talk

1. Argue that Einstein's cosmological constant Λ originally reared its ugly head with Newton, who used it (unsuccessfully) to calculate precession of the lunar apsides (perihelion advance of the moon).
2. Explain Newton's theory of revolving orbits and his approximation of precession for central force laws, valid for nearly circular orbits.
3. Extend Newton's method based on elliptical orbits to one based on general relativistic orbits (work in progress).

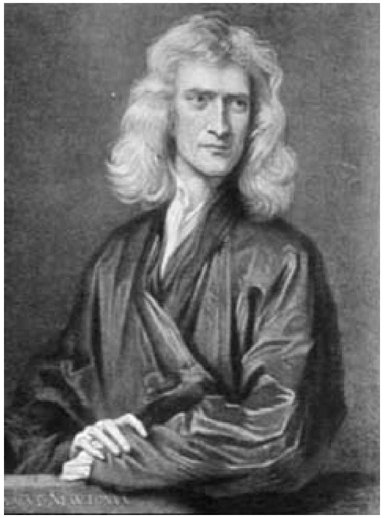
Gravitation and Celestial Mechanics

Newton (Principia, 1687)



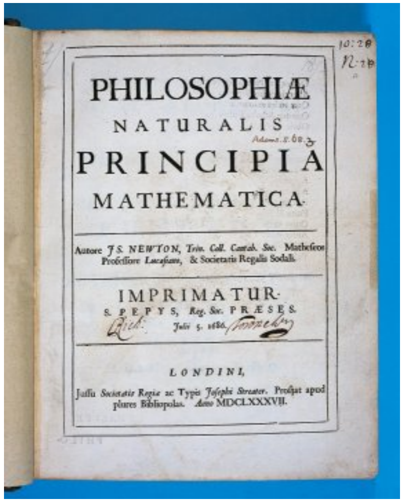
Gravity (inverse-square force)

$$F_N = -\frac{GMm}{r^2}, \qquad G = 6.67 \times 10^{-11}\ \mathrm{m^3/kg/s^2}$$



Sir Isaac Newton (1643-1727)

http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Newton.html



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

Title page of the Principia (1687)

http://www.lib.cam.ac.uk/Exhibitions/Footprints_of_the_Lion/gravity_glory.html

Planetary Motion (Two-Body Problem)

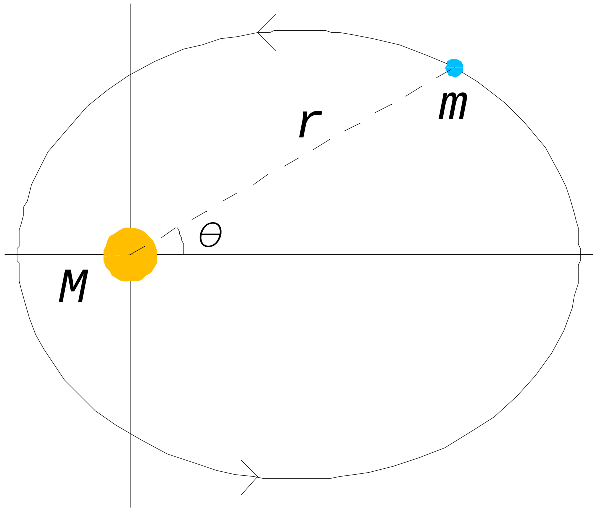
Newton's Second Law of Motion

$$F = ma$$

$$-\frac{GM}{r^2} = \frac{d^2 r}{dt^2} - r\omega^2 \qquad \omega = \frac{d\theta}{dt} \quad \text{(angular velocity)}$$

Conservation of Angular Momentum

$$h = r^2\omega = r^2\frac{d\theta}{dt} \quad \text{(constant)}$$



Kepler's Area Law

$$\frac{dA}{dt} = \frac{d}{dt}\left[\frac{1}{2}\int_\alpha^\beta r^2 d\theta\right] = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h$$

(Planets sweep out equal areas in equal times)

Equation of motion

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} + \frac{h^2}{r^3}$$

Linearization

1. Change of variables: $u = \frac{1}{r}, \quad \frac{d}{dt} \rightarrow \frac{d}{d\theta}$

$$h = r^2 \frac{d\theta}{dt} \Rightarrow \frac{dr}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{h}{r^2 u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$\Rightarrow \frac{d^2 r}{dt^2} = -h\frac{d^2 u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h^2 u^2 \frac{d^2 u}{d\theta^2}$$

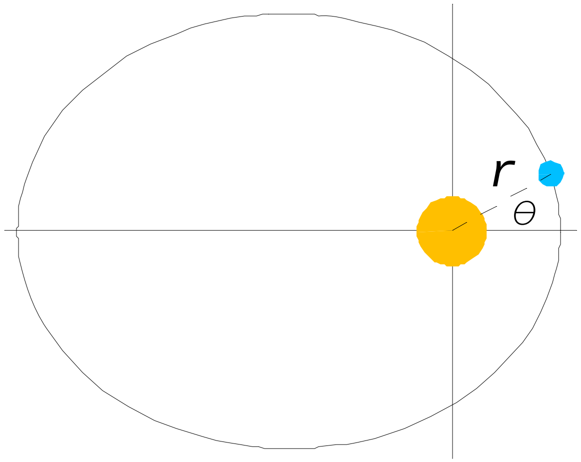
2. Linear 2nd-order differential equation

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} + \frac{h^2}{r^3} \Rightarrow \boxed{\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2}}$$

Solution (assume $r'(0) = 0$):

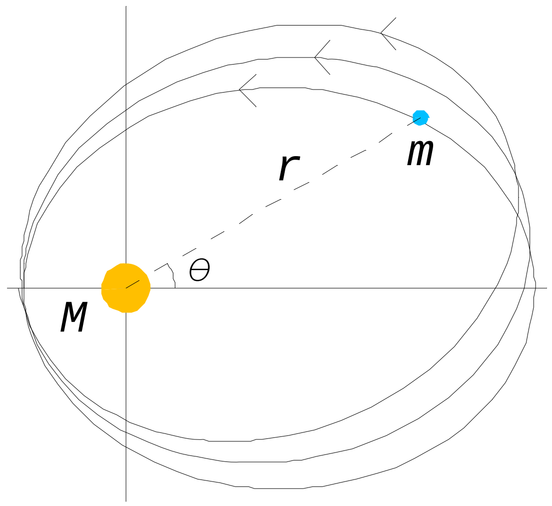
$$u = \frac{1}{r} = \frac{GM}{h^2} + A\cos\theta$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad (e = \text{eccentricity})$$



Elliptical orbit ($0 < e < 1$)

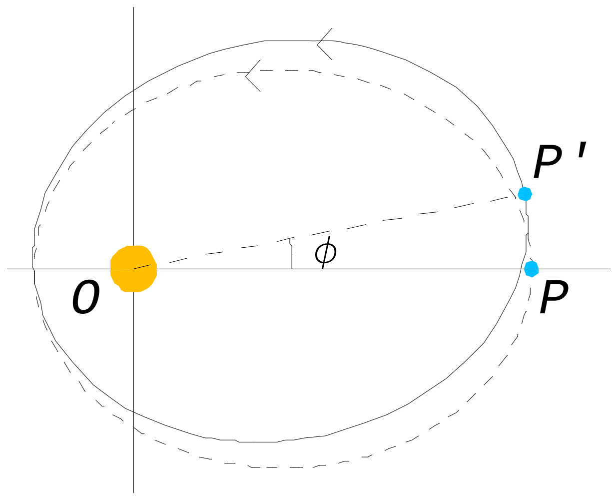
Precession of the Line of Apsides



Perihelion advance of Mercury:

http://zebu.uoregon.edu/images/precess.mpg

Line of Apsides:

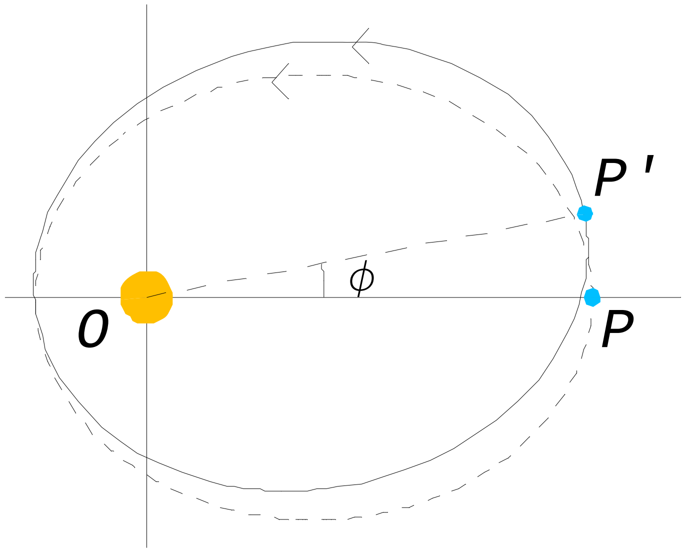


OP = Line of apside corresponding to aphelion (or perihelion) of orbit

ϕ = Precession of the apside *OP* (radians/revolution)

Revolving orbits

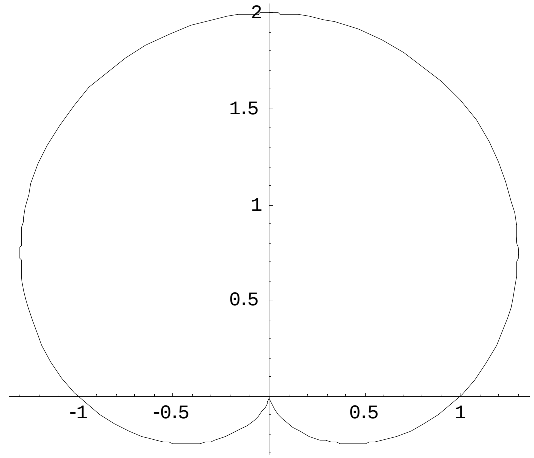
Approximate orbits under arbitrary central force laws by revolving elliptical orbits $r(\alpha\theta)$, where α is the rate of rotation.



"Revolving" elliptical orbit $r(\alpha\theta)$

$$\phi = 2\pi\alpha - 2\pi$$

```
In[2]:= r[t_] = 1 + Sin[t];
        PolarPlot[r[t], {t, -Pi/2, 3Pi/2}]
```




```
In[4]:= α = 1.05;
        PolarPlot[r[α * t], {t, -Pi/2, 2Pi + 3Pi/2}]
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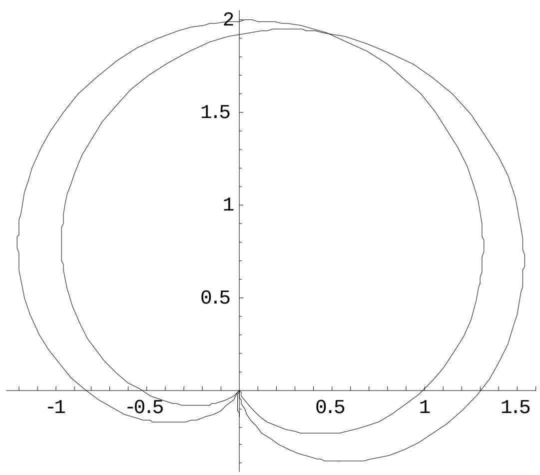
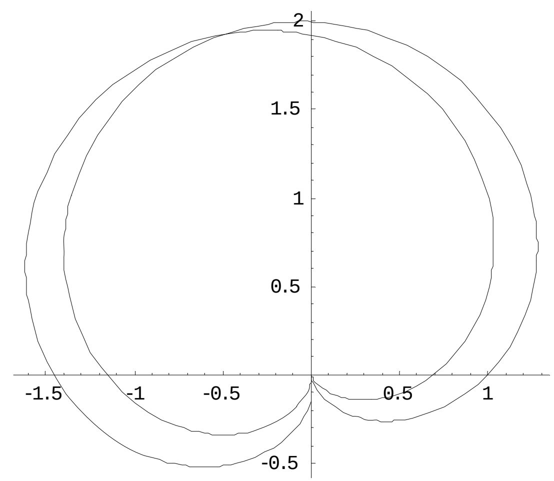


$\alpha > 1$ (clockwise)

```
In[6]:= α = 0.95;
        PolarPlot[r[α * t], {t, -Pi/2, 2Pi + 3Pi/2}]
```



$\alpha < 1$ (counter-clockwise)

Newton's Theorem on Revolving Orbits

What type of force law describes a revolving orbit?

Theorem: Let $r(\theta)$ be an orbit generated by any central force $F(r)$. Then the revolving orbit $r(\tilde{\theta}) = r(\alpha\theta)$ is generated by a central force $\tilde{F}(r)$ that differs from $F(r)$ by an inverse-cube force, and conversely. In particular, if h and $\tilde{h}$ are the angular momentums corresponding to $r(\theta)$ and $r(\tilde{\theta})$, respectively, then

I. $\tilde{F}(r) = F(r) + \dfrac{h^2 - \tilde{h}^2}{r^3}$

II. $\alpha = \dfrac{\tilde{h}}{h}$

Proof:

The equation of motion for $u(\theta) = 1/r(\theta)$, subject to a central force law $F(u)$, can be shown to satisfy

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(u)}{h^2u^2} \qquad (1)$$

It suffices to show that the equation of motion for $u(\tilde{\theta})$, where $\tilde{\theta} = \alpha\theta$ and subject to a central force law $\tilde{F}(u)$, also satisfies (1), and vice versa:

$$\Rightarrow: \ \frac{d^2u}{d\tilde{\theta}^2} + u = \frac{1}{\alpha^2}\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha^2}\left(\frac{d^2u}{d\theta^2} + u\right) + \left(1 - \frac{1}{\alpha^2}\right)u$$

$$= -\frac{1}{\alpha^2}\frac{F(u)}{h^2u^2} + \left(1 - \frac{1}{\alpha^2}\right)u = -\frac{1}{\alpha^2}\frac{F(u)}{h^2u^2} + \frac{(\alpha^2 - 1)h^2u^3}{\alpha^2h^2u^2}$$

$$= -\frac{F(u) + (h^2 - \tilde{h}^2)u^3}{\tilde{h}^2u^2} = -\frac{\tilde{F}(u)}{\tilde{h}^2u^2}, \qquad (\tilde{h} = \alpha h)$$

Precession of Apsides of Nearly Circular Orbits

"It is required to find the motions of the apsides of orbits that differ very little from circles" - Principia (Book I, Prop. 45)

Elliptical orbit

$$F_N(r) = -\frac{GM}{r^2} = -\frac{h^2}{Rr^2}$$

$$R = \frac{h^2}{GM} = a(1-e^2) \quad \text{(Semi-latus rectum of ellipse)}$$

Revolving elliptical orbit

$$\tilde{F}_N(r) = F_N(r) + \frac{h^2 - \tilde{h}^2}{r^3} = -\frac{h^2}{Rr^2} + \frac{h^2 - \tilde{h}^2}{r^3}$$

$$= \boxed{-\frac{h^2 r + R(\tilde{h}^2 - h^2)}{Rr^3}}$$

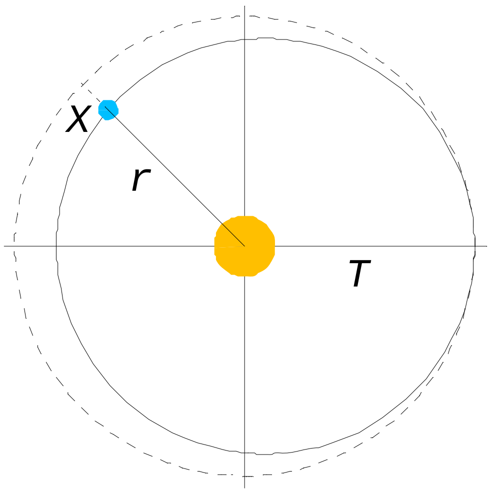
Nearly circular orbit under arbitrary central force $F(r)$

1. Assume orbit is nearly circular so that $r = T - X$, where T is the maximum radius and $X = T - r$ is relatively small.

$$F(r) = -\frac{C(r)}{Rr^3} = -\frac{C(T-X)}{Rr^3}$$

$$= -\frac{C(T) - C'(T)X + C''(T)X^2/2 - \ldots}{Rr^3}$$

$$\approx \boxed{-\frac{C(T) - C'(T)X}{Rr^3}} \quad \text{(Linear approximation)}$$



2. Identify the linear approximation of $F(r)$ as a revolving elliptical orbit.

$$F(r) \approx \tilde{F}_N(r) = -\frac{h^2 r + R(\tilde{h}^2 - h^2)}{Rr^3} = \boxed{-\frac{h^2(T-X) + R(\tilde{h}^2 - h^2)}{Rr^3}}$$

3. Equate numerators in Parts 1 and 2 and assume $R \approx T$ (semi-latus rectum ≈ maximum radius):

$$\begin{aligned} C(T) - C'(T)X &= h^2(T - X) + R(\tilde{h}^2 - h^2) \\ &\approx h^2(T - X) + T(\tilde{h}^2 - h^2) \\ &\approx \tilde{h}^2 T - h^2 X \end{aligned}$$

$$\Rightarrow \begin{cases} C(T) = \tilde{h}^2 T \\ C'(T) = h^2 \end{cases}$$

4. Newton's formula for precession:

$$\alpha^2 = \frac{\tilde{h}^2}{h^2} = \frac{C(T)}{TC'(T)} = \boxed{\left. \frac{C(T)}{TC'(T)} \right|_{T=1}}$$

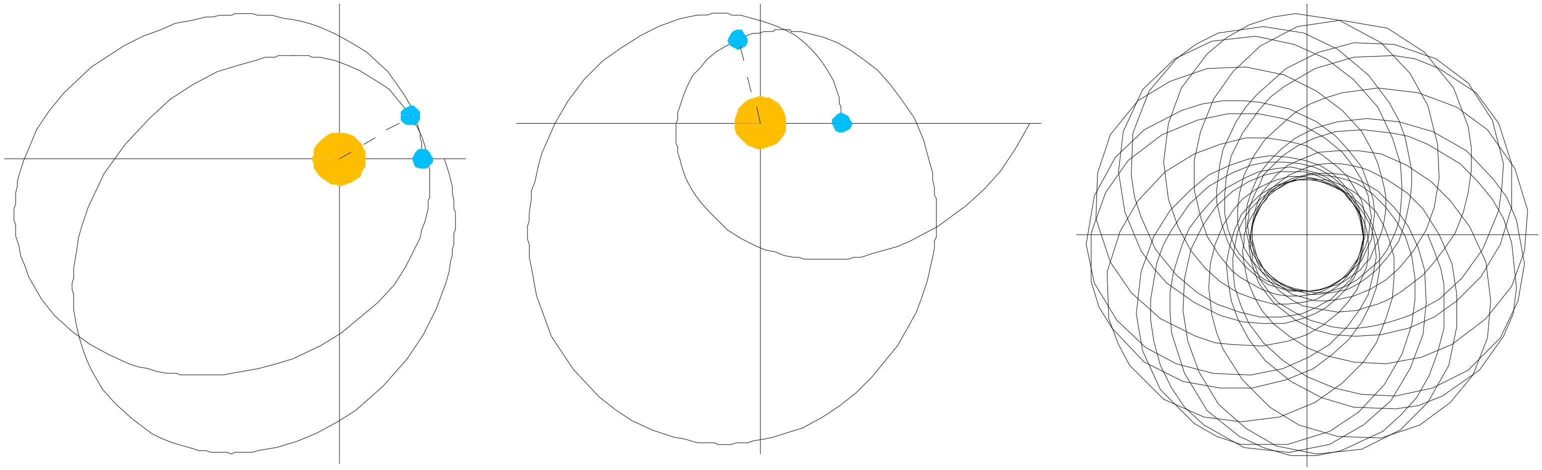
Note: The substitution $T = 1$ can be made after normalizing the orbit $r(\theta)$ appropriately.

Examples:

1. $C(r) = ar^n$

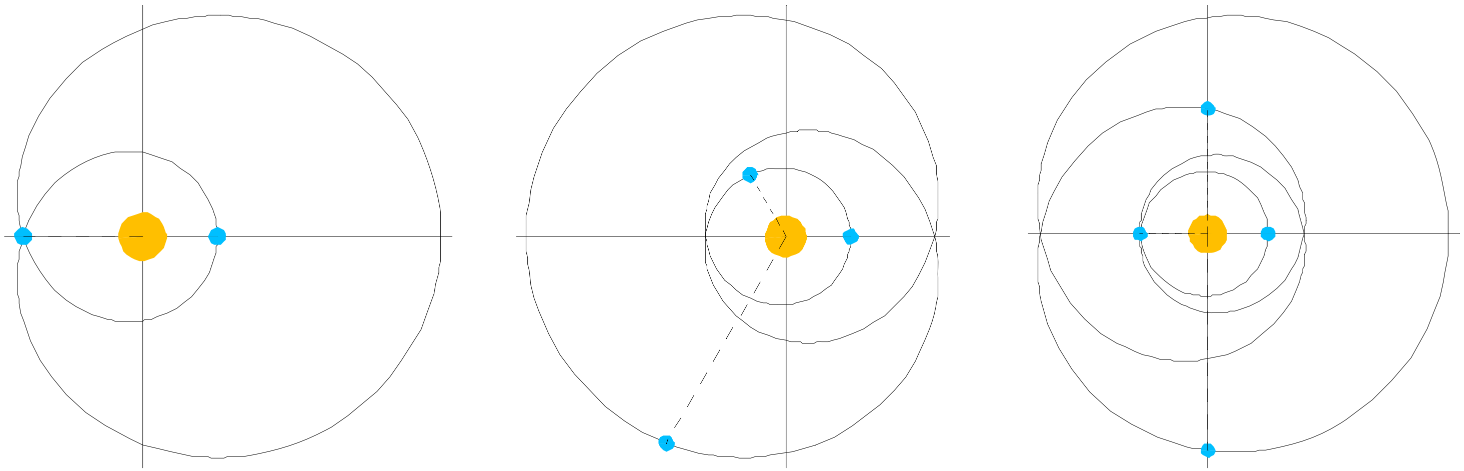
$$\alpha = \sqrt{\frac{C(T)}{TC'(T)}} = \sqrt{\frac{aT^n}{T(naT^{n-1})}} = \sqrt{\frac{1}{n}}$$

Note: Observe that the inverse square force law (n = 1) yields zero precession (α = 1).



$n = 1.2,\ \phi \approx 31^\circ$ $n = 2,\ \phi \approx 105^\circ$ $n = 2,$ 50 periods

Periodic orbits:



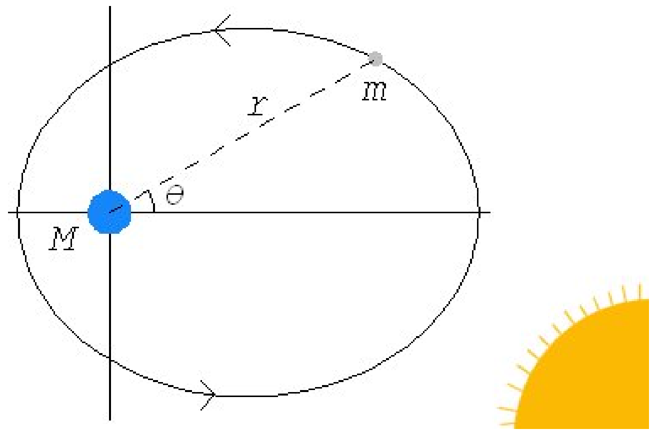
$n=4,\ \phi=180^\circ$ $\quad$ $n=9,\ \phi=240^\circ$ $\quad$ $n=16,\ \phi=270^\circ$

2. $C(r)=ar^m+br^n$

$$\alpha=\sqrt{\left.\frac{C(T)}{TC'(T)}\right|_{T=1}}=\sqrt{\left.\frac{aT^m+bT^n}{T(maT^{m-1}+nbT^{n-1})}\right|_{T=1}}=\sqrt{\frac{a+b}{am+bn}}$$

Precession of Lunar Apsides
(advance of lunar perigee)

Earth-Moon-Sun



Newton's 'foreign force' (Sun's gravity)

$$F(r) = -\frac{GM}{r^2} + kGMr \qquad \text{(Earth + Sun)}$$

k = Ratio of foreign force to Earth's gravity

$$\approx \frac{100}{35745}$$

Newton's calculation of precession:

$$F(r) = -\frac{GM}{r^2} + kGMr = -\frac{GM(r - kr^4)}{r^3}$$

$$= -\frac{h^2(r - kr^4)}{Rr^3} = -\frac{C(r)}{Rr^3}$$

$$C(r) = h^2 r - h^2 kr^4 \qquad (a = h^2, b = -h^2 k, m = 1, n = 4)$$

$$\alpha = \sqrt{\frac{C(T)}{TC'(T)}}\Bigg|_{T=1} = \sqrt{\frac{a+b}{am+bn}} = \sqrt{\frac{1-k}{1-4k}}\Bigg|_{k=\frac{100}{35745}}$$

$$= 1.00423$$

$$\therefore \phi = 360\alpha - 360 \approx \boxed{1.5} \text{ degrees/revolution}$$

Remarks:

1. Newton's answer for the precession of the lunar apside is incorrect. The observed value is approximately 3° per revolution (or month), or as he writes in the Principia,

   "*The apsis of the moon is about twice as swift*".

2. It is unclear how Newton obtains the value $k = 100/35745$. The correct answer, assuming Newton's foreign force, requires only roughly half this value.

3. Newton's formula for precession can also be used to calculate (correctly this time) the perihelion advance of Mercury, one of the hallmark tests of Einstein's theory of general relativity.

General Relativity

Mercury

- Additional 43 arcseconds/century unexplained by Newtonian theory (even accounting for the gravitational effects of all the other planets)

Gravity (inverse-square + inverse-quartic force)

$$F_E(r) = -\frac{GM}{r^2} - \frac{3GMh^2}{c^2 r^4} \qquad c = 3\times 10^8 \text{ m/s} \text{ (speed of light)}$$

$$= -\frac{(h^2 r + 3h^4 c^{-2} r^{-1})}{Rr^3} \qquad R = \frac{h^2}{GM} \approx a(1-e^2)$$

$$= -\frac{C(r)}{Rr^3} \qquad C(r) = h^2 r + 3h^4 c^{-2} r^{-1}$$

Precession due to GR: $(a = h^2,\ b = 3h^4c^{-2},\ m = 1,\ n = -1)$

$$\alpha = \sqrt{\frac{C(T)}{TC'(T)}}\Bigg|_{T=1} = \sqrt{\frac{a+b}{am+bn}} = \sqrt{\frac{1+3h^2c^{-2}}{1-3h^2c^{-2}}} \approx \sqrt{1+6h^2c^{-2}}$$

$$\approx 1+3h^2c^{-2} = 1+\frac{3GMc^{-2}}{R} \approx 1+\frac{3GMc^{-2}}{a(1-e^2)}$$

Precession of Mercury due to GR:

$$\phi = 2\pi(\alpha-1)$$

$$\approx \frac{6\pi GMc^{-2}}{a(1-e^2)}$$

$$\approx \frac{6\pi(6.673\times10^{-11})(1.99\times10^{30})(2.9979\times10^{8})^{-2}}{5.79\times10^{10}(1-0.2056^2)}$$

$$\approx 5.0224\times10^{-7}\ \text{radians/revolution}$$

$$\approx 43\ \text{arcseconds/century}$$

Who Blundered First?

GR + Cosmological constant Λ

$$F = F_E + \frac{\Lambda}{3} r = -\frac{GM}{r^2} - \frac{3GMh^2}{c^2 r^4} + \frac{\Lambda}{3} r$$

Newton versus Einstein

Newton + k

$$F = -\frac{GM}{r^2} + kGMr$$

Einstein + Λ

$$F = -\frac{GM}{r^2} - \frac{3GMh^2}{c^2 r^4} + \frac{\Lambda}{3} r$$

Conclusion

Foreign force = Cosmological constant

Second-Order Approximation of Precession

General relativistic (GR) orbit

$$F_E(r) = -\frac{GM}{r^2} - \frac{3GMh^2}{c^2 r^4} = -\frac{h^2 r^2 + 3h^4 c^{-2}}{R r^4} \qquad R = \frac{h^2}{GM}$$

Revolving GR orbit

$$\tilde{F}(r) = F_E(r) + \frac{h^2 - \tilde{h}^2}{r^3} = -\frac{h^2 c^2 r^2 + 3h^4}{R c^2 r^4} + \frac{h^2 - \tilde{h}^2}{r^3}$$

$$= \boxed{-\frac{h^2 r^2 + 3h^4 c^{-2} + R(\tilde{h}^2 - h^2) r}{R r^4}}$$

Nearly circular orbit under arbitrary central force $F(r)$

1. Assume orbit is nearly circular so that $r = T - X$, where T is the maximum radius and $X = T - r$ is relatively small.

$$F(r) = -\frac{C(r)}{Rr^4} \approx \boxed{-\frac{C(T) - C'(T)X + C''(T)X^2/2}{Rr^4}}$$

(Quadratic approximation)

2. Identify the quadratic approximation of $F(r)$ as a revolving GR orbit.

$$F(r) \approx \tilde{F}(r)$$

$$\approx \boxed{-\frac{h^2(T-X)^2 + 3h^4c^{-2} + R(\tilde{h}^2 - h^2)(T-X)}{Rr^4}}$$

3. Equate numerators in Parts 1 and 2 and again assume $R \approx T$:

$$
\begin{aligned}
C(T) - C'(T)X + C''(T)X^2/2 & \\
&= h^2(T-X)^2 + 3h^4c^{-2} + R(\tilde{h}^2 - h^2)(T-X) \\
&\approx \tilde{h}^2T^2 + 3h^4c^{-2} - (\tilde{h}^2 + h^2)TX + h^2X^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow C(T) &= \tilde{h}^2T^2 + 3h^4c^{-2} \\
C'(T) &= (\tilde{h}^2 + h^2)T \\
C''(T) &= 2h^2
\end{aligned}
$$

4. Formula for precession based on revolving GR orbit (based on the first and second equations):

$$
\alpha^2 = \frac{\tilde{h}^2}{h^2} = \frac{6C'(T) - c^2T^3 \mp cT\sqrt{12C(T) - 12TC'(T) + c^2T^4}}{c^2T^3 \pm cT\sqrt{12C(T) - 12TC'(T) + c^2T^4}}
$$

Classical Limit ($c \to \infty$) (?)

$$\lim_{c\to\infty} \alpha^2 = \frac{C(T)}{TC'(T) - C(T)}$$

References

[1] S. Chandrasekhar, Newton's Principia for the Common Reader, Oxford University Press, 1995.

[2] Alan Cook, Success and failure in Newton's lunar theory, Astronomy & Geophysics 41 (2000), No. 6, 21-25.

[3] Isaac Newton, The Principia: Mathematical Principles of Natural Philosophy, A New Translation by I. Bernard Cohen and Anne Whitman, University of California Press, 1999.

[4] Peter Rowlands, A simple approach to the experimental consequences of general relativity, Phys. Educ. **32** (1997), 49-55.

[5] Blake Temple and Craig A. Tracy, From Newton to Einstein, The American Mathematical Monthly 99 (1992), No. 6, 507-521.

[6] Eric W. Weisstein, Two-Body Problem, Eric Weisstein's World of Physics, http://scienceworld.wolfram.com/physics/Two-BodyProblem.html.