Pascal's Square:
Determinants, Bernoulli Polynomials, and the Arithmetical Triangle

Robert Booth and Hieu D. Nguyen*

Rowan University
Glassboro, NJ

MathFest
San Jose, CA
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Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

...

Binomial Expansion:

$$(1+x)^0 = 1$$

$$(1+x)^1 = 1+x$$

$$(1+x)^2 = 1+2x+x^2$$

$$(1+x)^3 = 1+3x+3x^2+x^3$$

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^2+...+\binom{n}{k}x^k+...+x^n$$

Arithmetical Triangle (Pascal, 1654)

1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1				
...				

Figurate Triangle (Jacob Bernoulli, 1713)

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1
...				

Pascal's Square

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 3 & 1 & 0 & \dots \\ 1 & 4 & 6 & 4 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} = (p_{mn})$$

$$p_{mn} = \binom{m-1}{n-1}$$

(Recall that $\binom{i}{j} = 0$ for $i < j$)

Bernoulli Polynomials

Sums of powers:

$$\sum_{n=1}^{N} n^p = \sum_{n=0}^{p} (-1)^{\delta_{np}} \frac{p!}{n!(p+1-n)!} B_n N^{p+1-n}$$

Bernoulli numbers:

$$B_0 = 1, \qquad B_1 = 1/2 \text{ (or } -1/2),$$
$$B_2 = 1/6, \qquad B_3 = 0,$$
$$B_4 = -1/30, \qquad B_5 = 0.$$

Generating function (Euler):

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

Bernoulli polynomials:

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \qquad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x,$$

...

$$B_n = B_n(0)$$

Quotients of Power Series

$$\frac{f(t)}{g(t)} = \frac{\sum_{n=0}^{\infty} c_n t^n}{\sum_{n=0}^{\infty} a_n t^n} = \sum_{n=0}^{\infty} A_n t^n$$

$$\begin{aligned} c_0 &= a_0 A_0, \\ c_1 &= a_0 A_1 + a_1 A_0, \\ c_2 &= a_0 A_2 + a_1 A_1 + a_2 A_0, \\ &\ldots \\ c_n &= a_0 A_n + a_1 A_{n-1} + \ldots + a_n A_0. \end{aligned}$$

Cramer's Rule:

$$A_n = (-1)^n \frac{1}{a_0^n} \begin{vmatrix} c_0 & a_0 & 0 & 0 & \ldots & 0 \\ c_1 & a_1 & a_0 & 0 & \ldots & 0 \\ c_2 & a_2 & a_1 & a_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n-1} & a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \\ c_n & a_n & a_{n-1} & a_{n-2} & \ldots & a_1 \end{vmatrix}_{n+1}$$

Determinant Formula for Bernoulli Polynomials

$$\frac{e^{xt}}{\frac{e^t-1}{t}} = \frac{\sum_{n=0}^{\infty}\frac{x^n t^n}{n!}}{\sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n$$

$$c_n = x^n / n!$$

$$a_n = 1/(n+1)!$$

$$A_n = B_n(x)/n!$$

$$B_n(x) = n!(-1)^{(n)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \frac{x}{1!} & \frac{1}{2!} & 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{x^2}{2!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \ldots & 0 \\ \frac{x^3}{3!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{x^n}{n!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \ldots & 1 \end{vmatrix}_{n+1}$$

Elementary Row and Column Operations

<u>Step 1:</u> Factor $1/(i-1)!$ from row i for $i = 1, 2, \ldots, n+1$.

$$B_n(x) = \frac{n!(-1)^{(n)}}{1!2!3!\ldots(n-1)!(n)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ x & \frac{1}{2!} & 1 & 0 & 0 & 0 & 0 \\ x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & 0 \\ x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & 3! & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!} & \ldots & n! \end{vmatrix}_{n+1}$$

Step 2: Factor $(j - 3)!$ from row j for $j = 3, 4, \ldots, n+1$.

$$B_n(x) = \frac{(-1)^{(n)} 1!2!3!\ldots(n-2)!}{1!2!3!\ldots(n-1)!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ x & \frac{1}{2!} & 1 & 0 & 0 & 0 & 0 \\ x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & 0 \\ x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & \frac{3!}{2!} & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!2!} & \ldots & \frac{n!}{2!(n-2)!} \end{vmatrix}_{n+1}$$

Theorem: (Costabile–Dell'Accio–Gualtieri, 2006)

$$B_n(x) = \frac{(-1)^{(n)}}{(n-1)!} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ x & 1/2 & 1 & 0 & 0 & 0 & \dots & 0 \\ x^2 & 1/3 & 1 & 2 & 0 & 0 & \dots & 0 \\ x^3 & 1/4 & 1 & 3 & 3 & 0 & \dots & 0 \\ x^4 & 1/5 & 1 & 4 & 6 & 4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x^n & 1/(n+1) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} \end{vmatrix}_{n+1}$$

Observe that Pascal's square appears in red inside the matrix above with its main diagonal entries replaced by zeros.

Corollary: (Turnbull, 1960)

$$B_n = B_n(0) = \frac{(-1)^{n-1}}{(n+1)!} \begin{vmatrix} 1/2 & 1 & 0 & 0 & \ldots & 0 \\ 1/3 & 1 & 2 & 0 & \ldots & 0 \\ 1/4 & 1 & 3 & 3 & \ldots & 0 \\ 1/5 & 1 & 4 & 6 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n+1} & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \ldots & \binom{n}{n-2} \end{vmatrix}_n$$

Hypergeometric Bernoulli Polynomials

Howard (1977):

$$\frac{t^N e^{xt}/N!}{e^t - [1 + t + t^2/2! + \ldots + t^{N-1}/(N-1)!]} = \frac{e^{xt}}{{}_1F_1(1, N+1, t)} = \sum_{n=0}^{\infty} B_n(N,x)\frac{t^n}{n!}$$

$N = 0$: $$\frac{e^{xt}}{e^t} = \sum_{n=0}^{\infty} B_n(0,x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} (x-1)^n \frac{t^n}{n!}$$

$N = 1$: $$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(1,x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

$N = 2$: $$\frac{t^2 e^{xt}/2!}{e^t - (1+t)} = \sum_{n=0}^{\infty} B_n(2,x)\frac{t^n}{n!}$$

$N = 3$: $$\frac{t^3 e^{xt}/3!}{e^t - (1+t+t^2/2!)} = \sum_{n=0}^{\infty} B_n(3,x)\frac{t^n}{n!}$$

Theorem: $(N = 0)$

$$B_n(0,x) = (-1)^{(n)} \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ x & 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ x^2 & 1 & 2 & 1 & 0 & 0 & \dots & 0 \\ x^3 & 1 & 3 & 3 & 1 & 0 & \dots & 0 \\ x^4 & 1 & 4 & 6 & 4 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ x^n & 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \dots & \binom{n}{n-1} \end{vmatrix}_{n+1} = (x-1)^n$$

Observe that Pascal's square appears (in red) inside the matrix above.

Theorem: $(N = 2)$

$$B_n(2,x) = \frac{(-1)^{(n)}2^{(n-1)}}{(n-1)!(n-2)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ x & \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 \\ x^2 & \frac{1}{6} & \frac{1}{3} & 1 & 0 & 0 & \cdots & 0 \\ x^3 & \frac{1}{10} & \frac{1}{4} & 1 & 3 & 0 & \cdots & 0 \\ x^4 & \frac{1}{15} & \frac{1}{5} & 1 & 4 & 6 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x^n & \frac{1}{\binom{n+2}{2}} & \frac{1}{\binom{n+1}{1}} & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-3} \end{vmatrix}_{n+1}$$

Observe that Pascal's square appears (in red) with its main and first lower off-diagonal entries replaced by zeros.

Theorem: (N arbitrary positive integer)

$$B_n(N,x) = \frac{(-1)^{(n)}(N!)^n 1!2!3!\ldots(n-N-1)!}{1!2!3!\ldots(n-1)!1!2!3!\ldots N!}\left|b_{ij}\right|$$

$$b_{ij} = \begin{cases} x^{i-1} & j = 1 \\ \binom{i-j+N+1}{i-1}^{-1} & 2 \le j \le N+2 \\ \binom{i-1}{j-N-2} & j \ge N+2 \end{cases}$$

Note: Pascal's square appears inside the matrix (b_{ij}) with its main and first N – 1 lower off-diagonal entries replaced by zeros.

References

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