The Search for Nothing

A zero-free region of hypergeometric zeta

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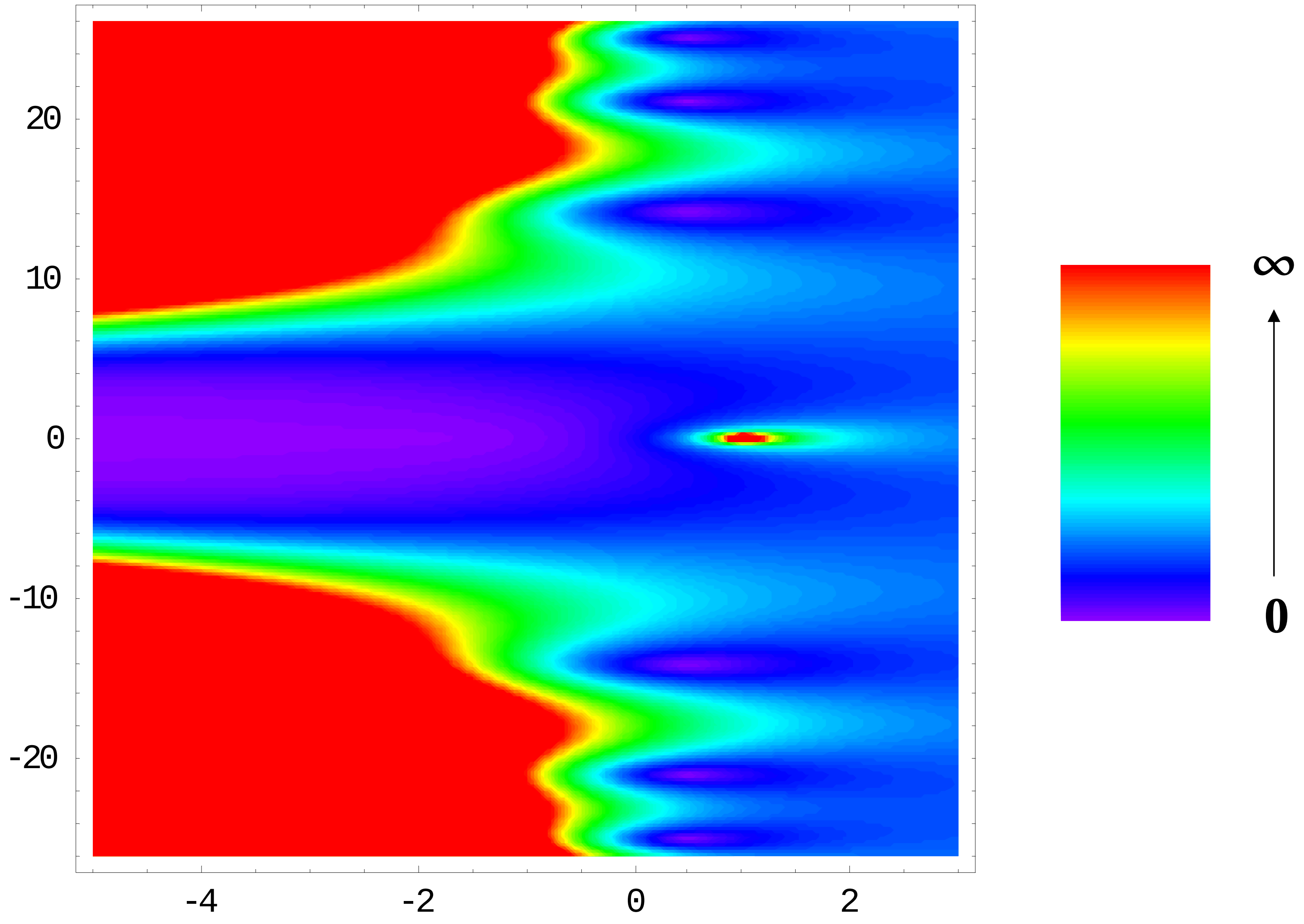


20
10
0
-10
-20
-4
-2
0
2
∞
0

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \qquad (\mathrm{Re}(s) > 1)$$

$$s = \sigma + it \qquad \text{(Complex variable)}$$

Leonhard Euler

Born: 15 April 1707 in Basel, Switzerland
Died: 18 Sept 1783 in St Petersburg, Russia

Georg Friedrich Bernhard Riemann

Born: 17 Sept 1826 in Breselenz, Hanover (now Germany)
Died: 20 July 1866 in Selasca, Italy

http://www-gap.dcs.st-and.ac.uk/~history/

Analytic Continuation

Theorem: (Euler/Riemann) $\zeta(s)$ is analytic for all complex values of s, except for a simple pole at $s = 1$.

Special Values of Zeta:

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots = \infty$$

Diverges (N. d'Oresme, 1323-82)

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

Basel Problem (L. Euler, 1735)

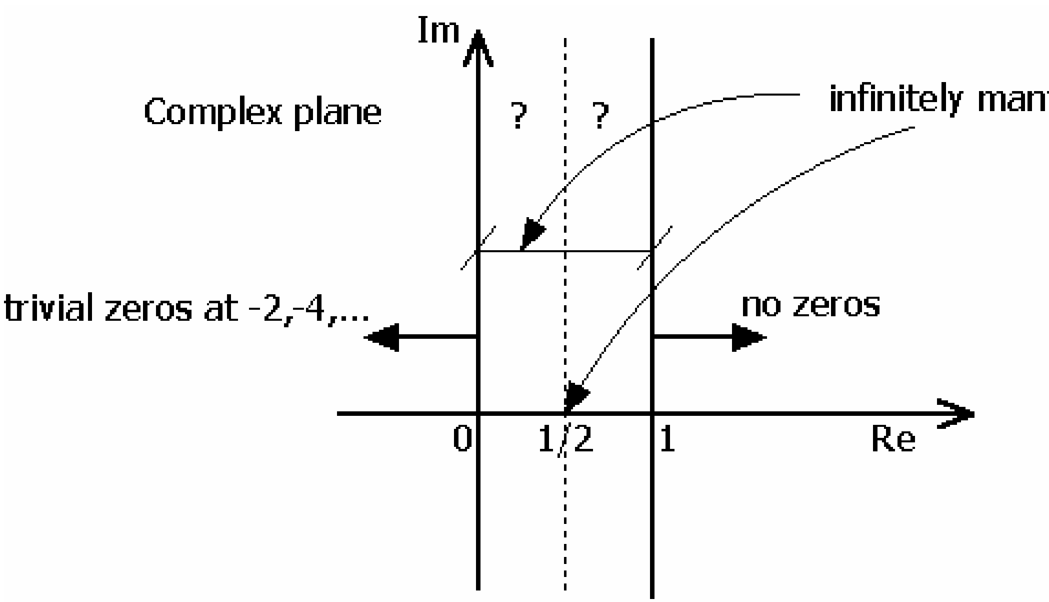
$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots = ?$$

Irrational (R. Apery, 1978)

Zeros of Zeta

Trivial Zeros: $\zeta(-2) = \zeta(-4) = \zeta(-6) = \ldots = 0$

Riemann Hypothesis (Millennium Prize Problem):
The nontrivial zeros of $\zeta(s)$ are all located on the 'critical line' $\mathrm{Re}(s) = 1/2$.



http://users.forthnet.gr/ath/kimon/Riemann/Riemann.htm

http://mathworld.wolfram.com/RiemannZetaFunction.html

Zero-Free Regions of Zeta

Theorem: (E/R) $\zeta(s) \neq 0$ on the right half-plane $\mathrm{Re}(s) > 1$.

Proof: Follows from Euler's product formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\mathrm{Re}(s) > 1)$$

$$\neq 0$$

Theorem: (E/R) $\zeta(s) \neq 0$ on the left half-plane $\mathrm{Re}(s) < 0$, except for trivial zeros at $s = -2, -4, -6, \ldots$.

Proof: Follows from the functional equation (reflection about the critical line $\mathrm{Re}(s) = 1/2$):

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Hypergeometric Zeta Functions

$$\zeta_N(s) \equiv \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)}\, dx \qquad (N \in \mathbb{N})$$

where

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^N}{N!}$$

Cases:

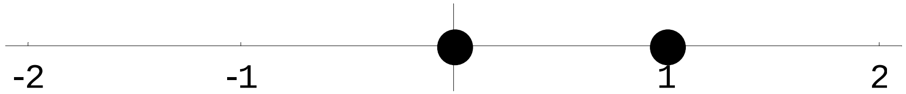
$$N = 1: \qquad \zeta_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \qquad \text{(Classical zeta)}$$

$$N = 2: \qquad \zeta_2(s) = \frac{1}{\Gamma(s+1)} \int_0^\infty \frac{x^{s}}{e^x - 1 - x}\, dx$$

$$N = 3: \qquad \zeta_3(s) = \frac{1}{\Gamma(s+2)} \int_0^\infty \frac{x^{s+1}}{e^x - 1 - x - x^2/2}\, dx$$

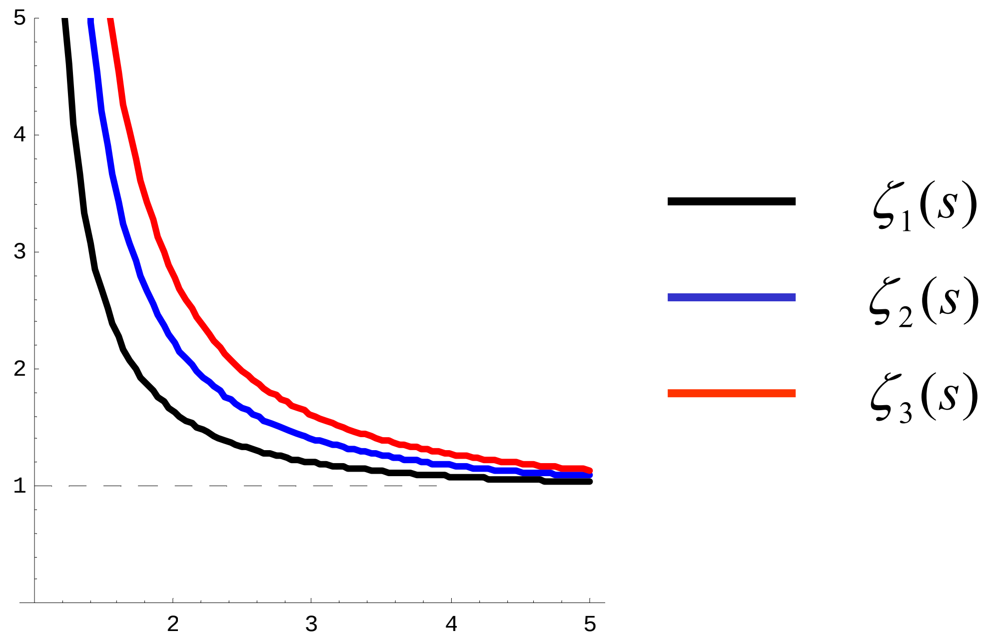
Theorem: $\zeta_N(s)$ is analytic for all complex values of s, except for N simple poles at $s = 2 - N, 3 - N, \ldots, 0, 1$.

$(N = 2)$



Theorem: For $N > 1$ and real $s > 1$,

$$\zeta_N(s) > \zeta_1(s)$$



Pre-Functional Equation

Theorem: For $\mathrm{Re}(s) < 0$,

$$\zeta_N(s) = 2\Gamma(1-(s+N-1))\sum_{n=1}^{\infty} r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]$$

Here $z_n = r_n e^{i\theta_n}$ are the roots of $e^z - T_{N\text{-}1}(z) = 0$ in the upper-half complex plane.

$N = 1$: The roots of $e^z - 1 = 0$ are given by $z_n = r_n e^{i\theta_n} = 2n\pi i$.

$$\zeta_1(s) = 2\Gamma(1-s)\sum_{n=1}^{\infty}(2n\pi)^{s-1}\cos[(s-1)(\pi-\pi/2)]$$

$$= 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1}$$

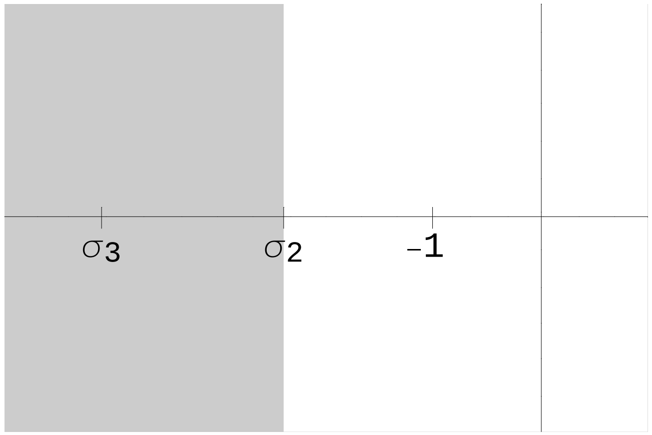
$$= 2(2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta_1(1-s) \qquad \text{(Functional equation)}$$

Zero-Free Region of Hypergeometric Zeta

Main Theorem: $\zeta_2(s) \neq 0$ on the left half-plane $\{\mathrm{Re}(s) < \sigma_2\}$, except for infinitely many trivial zeros located on the negative real axis, one in each of the intervals $S_m = [\sigma_{m+1}, \sigma_m]$, $m \geq 2$, where

$$\sigma_m = 1 - \frac{m\pi}{\pi - \theta_1}$$

$$\sigma_2 \approx -2.40781$$



Proof (?):

- No product formula
- No functional equation
- No swift proof

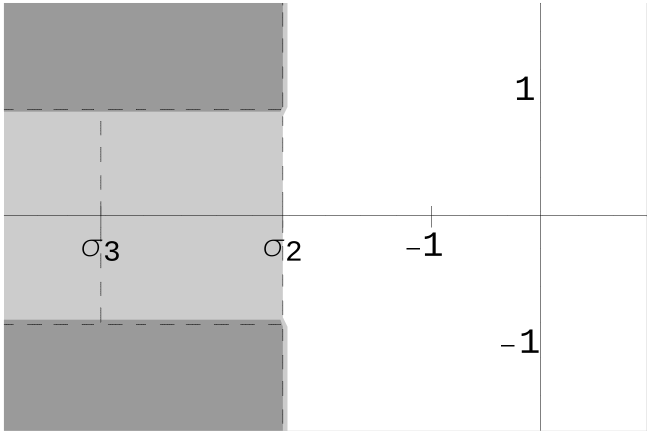
Sketch of Proof (Spira's Method)

$$\sum_{n=1}^{\infty} a_n = a_1 + \sum_{n=2}^{\infty} a_n$$

I. Key ingredient: Pre-functional equation for $\zeta_N(s)$:

$$\zeta_N(s) = 2\Gamma(1-(s+N-1))\sum_{n=1}^{\infty} r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]$$

II. Divide $\mathrm{Re}(s) < \sigma_2$ and conquer:



III. Demonstrate that pre-functional equation is dominated by the first term (n = 1) in each region and apply Rouche's theorem to compare roots.

Roots of $e^z - 1 - z = 0$ $(N = 2)$

Lemma: (Howard) The roots $\{z_n = x_n + iy_n\}$ of $e^z - 1 - z = 0$ that are located in the upper-half complex plane can be arranged in increasing order of modulus and argument:

$$|z_1| < |z_2| < |z_3| < \ldots$$
$$|\theta_1| < |\theta_2| < |\theta_3| < \ldots$$

Moreover, their imaginary parts satisfy the bound

$$(2n + 1/4)\pi < y_n < (2n + 1/2)\pi$$

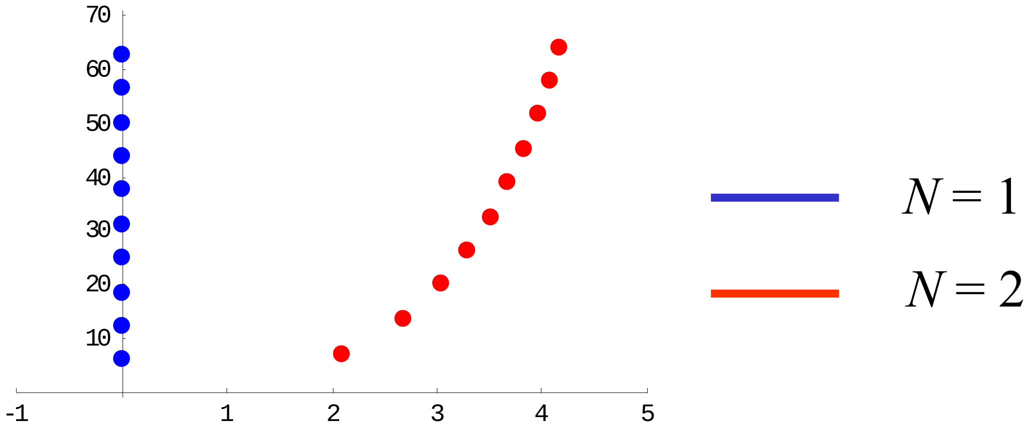


Table of Roots of $e^z - 1 - z = 0$ $(N = 2)$

{n	x_n	y_n	$(2n+1/4)\pi$	$(2n+1/2)\pi$	r_n	θ_n }
{1	2.0888	7.4615	7.069	7.854	7.7484	1.2978}
{2	2.6641	13.879	13.352	14.137	14.132	1.3812}
{3	3.0263	20.224	19.635	20.420	20.449	1.4223}
{4	3.2917	26.543	25.918	26.704	26.747	1.4474}
{5	3.5013	32.851	32.201	32.987	33.037	1.4646}
{6	3.6745	39.151	38.485	39.270	39.323	1.4772}
{7	3.8222	45.447	44.768	45.553	45.608	1.4869}
{8	3.9508	51.741	51.05	51.84	51.892	1.4946}
{9	4.0648	58.032	57.33	58.12	58.175	1.5009}
{10	4.1671	64.322	63.62	64.40	64.457	1.5061}

Theorem 1: (Dark Region) Let $s = \sigma + it$. If $\sigma < \sigma_2$ and $|t| > 1$, then $\zeta_2(s) \neq 0$.

Proof: We demonstrate that the pre-functional equation is dominated by the first term. Write

$$\zeta_2(s) = 2\Gamma(-s)I(s)$$

where

$$I(s) = \sum_{n=1}^{\infty} r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]$$

$$= r_1^{s-1}\cos[(s-1)(\pi-\theta_1)] + \sum_{n=2}^{\infty} r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]$$

$$= \overbrace{r_1^{s-1}\cos[(s-1)(\pi-\theta_1)]}^{f(s)}\left(1+\overbrace{\sum_{n=2}^{\infty}\frac{r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]}{r_1^{s-1}\cos[(s-1)(\pi-\theta_1)]}}^{g(s)}\right)$$

$$= f(s)(1+g(s))$$

Assuming $|g(s)| < 1$, it follows from the reverse triangle inequality that

$$\begin{aligned} |I(s)| &= |f(s)(1+g(s))| \\ &\geq |f(s)|\left(1-|g(s)|\right) \\ &> 0 \end{aligned}$$

Therefore, $I(s) \neq 0$ and hence $\zeta_2(s) \neq 0$ as desired.

Lemma: $|g(s)| < 1$.

Proof: We employ the following bounds:

1. Observe that

(i) $|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y$

(ii) $|\sinh y| \leq |\cos(x+iy)| \leq \cosh y$

It follows that for $|t| > 1$, we have the bound

$$\left|\frac{\cos[(s-1)(\pi-\theta_n)]}{\cos[(s-1)(\pi-\theta_1)]}\right| \le \left|\frac{\cosh[t(\pi-\theta_n)]}{\sinh[t(\pi-\theta_1)]}\right|$$
$$\le \left|\frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)}\right| \approx 0.971589$$
$$< 1$$

2. The roots r_n for $n > 1$ can be bounded by

$$r_n \ge \begin{cases} 3\pi m & \text{if } n = 2m-1 \text{ (odd)} \\ 4\pi m & \text{if } n = 2m \text{ (even)} \end{cases}$$

{m	r_{2m-1}	$3m\pi$	r_{2m}	$4m\pi$ }
{1	7.7484	--------	14.132	12.566}
{2	20.449	18.850	26.747	25.133}
{3	33.037	28.274	39.323	37.699}
{4	45.608	37.699	51.892	50.265}
{5	58.175	47.124	64.457	62.832}

It follows that

$$\left|g(s)\right| = \left|\sum_{n=2}^{\infty} \frac{r_n^{s-1}\cos[(s-1)(\pi-\theta_n)]}{r_1^{s-1}\cos[(s-1)(\pi-\theta_1)]}\right| \le \sum_{n=2}^{\infty}\left(\frac{r_n}{r_1}\right)^{\sigma-1}\left|\frac{\cos[(s-1)(\pi-\theta_n)]}{\cos[(s-1)(\pi-\theta_1)]}\right|$$

$$\le \sum_{n=2}^{\infty}\left(\frac{r_n}{r_1}\right)^{\sigma-1} \le \left[\sum_{m=1}^{\infty}\left(\frac{r_1}{4\pi m}\right)^{1-\sigma} + \sum_{m=2}^{\infty}\left(\frac{r_1}{3\pi m}\right)^{1-\sigma}\right]$$

$$\le \left[\left\{\left(\frac{r_1}{4\pi}\right)^{1-\sigma}\sum_{m=1}^{\infty}\frac{1}{m^{1-\sigma}} + \left(\frac{r_1}{3\pi}\right)^{1-\sigma}\sum_{m=1}^{\infty}\frac{1}{m^{1-\sigma}}\right\} - \left(\frac{r_1}{3\pi}\right)^{1-\sigma}\right]$$

$$\le \left[\left\{\left(\frac{r_1}{4\pi}\right)^{1-\sigma} + \left(\frac{r_1}{3\pi}\right)^{1-\sigma}\right\}\zeta(1-\sigma) - \left(\frac{r_1}{3\pi}\right)^{1-\sigma}\right]$$

$$\le \phi(\sigma)$$

Now, $\phi(\sigma) \le \phi(\sigma_2) \approx 0.2896 < 1$ since ϕ is increasing on $(-\infty, 0)$. Hence, $|g(s)| < 1$ as desired.

Theorem 2: (Light Region) Let $s = \sigma + it$. If $\sigma \leq \sigma_2$ and $|t| \leq 1$, then $\zeta_2(s) \neq 0$, except for infinitely many trivial zeros on the negative real axis, one in each of the intervals $S_m = [\sigma_{m+1}, \sigma_m]$, $m \geq 2$, where

$$\sigma_m = 1 - \frac{m\pi}{\pi - \theta_1}$$

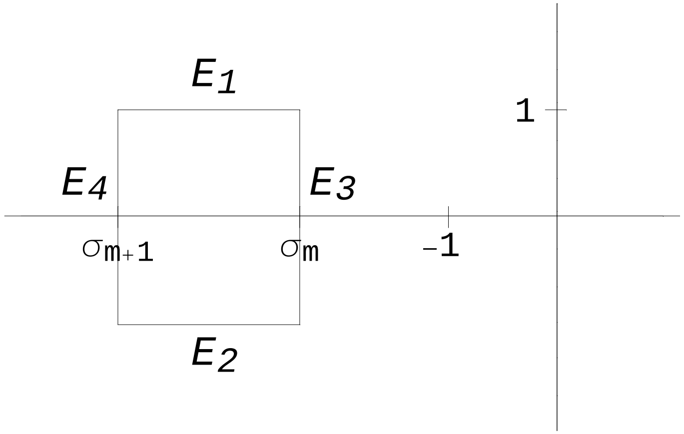
Proof: Define R_m to be the rectangle with edges E_1, E_2, E_3, and E_4:

$$E_1 = \{\sigma + i : \sigma_{m+1} \leq \sigma \leq \sigma_m\}$$

$$E_2 = \{\sigma - i : \sigma_{m+1} \leq \sigma \leq \sigma_m\}$$

$$E_3 = \{\sigma_m + it : |t| \leq 1\}$$

$$E_4 = \{\sigma_{m+1} + it : |t| \leq 1\}$$



We will demonstrate that $|g(s)| < 1$ on R_m so that

$$\left|I(s) - f(s)\right| = \left|f(s)g(s)\right| < \left|f(s)\right|$$

Then by Rouche's Theorem, $I(s)$ and $f(s)$ must have The same number of zeros inside R_m. Since the roots of $f(s)$ are none other than those of $\cos[(s\text{-}1)(\pi - \theta_1)]$, which has exactly one root in R_m, this proves that $I(s)$ also has exactly one root u_m in R_m. However,

$$I(\bar{u}_m) = \overline{I(u_m)}$$

This implies that $\bar{u}_m$ is also a root. Therefore, $u_m = \bar{u}_m$ and so u_m must be real and hence lies in the interval $[\sigma_{m+1}, \sigma_m]$. This proves the theorem.

Lemma: $|g(s)| < 1$ for all $s \in R_m$.

Proof: On E_1 we have

$$\left|\frac{\cos[(s-1)(\pi-\theta_n)]}{\cos[(s-1)(\pi-\theta_1)]}\right| \le \left[\frac{\cos^2\left[(\sigma-1)(\pi-\theta_n)\right]+\sinh^2(\pi-\theta_n)}{\cos^2\left[(\sigma-1)(\pi-\theta_1)\right]+\sinh^2(\pi-\theta_1)}\right]^{1/2}$$

$$\le \left[\frac{1+\sinh^2(\pi-\theta_n)}{\sinh^2(\pi-\theta_1)}\right]^{1/2} = \left|\frac{\cosh(\pi-\theta_n)}{\sinh(\pi-\theta_1)}\right|$$

$$\le \left|\frac{\cosh(\pi-\theta_2)}{\sinh(\pi-\theta_1)}\right| \approx 0.97158875$$

$$< 1$$

A similar argument holds for E_2.

As for E_3, we have

$$\left|\frac{\cos[(s-1)(\pi-\theta_n)]}{\cos[(s-1)(\pi-\theta_1)]}\right| \le \left[\frac{\cos^2\left[m\pi(\pi-\theta_n)/(\pi-\theta_1)\right]+\sinh^2[t(\pi-\theta_n)]}{\cos^2(-m\pi)+\sinh^2[t(\pi-\theta_1)]}\right]^{1/2}$$

$$\le \left[\frac{1+\sinh^2[t(\pi-\theta_n)]}{1+\sinh^2[t(\pi-\theta_1)]}\right]^{1/2} = \left|\frac{\cosh[t(\pi-\theta_n)]}{\cosh[t(\pi-\theta_1)]}\right|$$

$$\le 1$$

A similar argument holds for E_4. Therefore, on R_m,

$$|g(s)| \le \sum_{n=2}^{\infty}\left(\frac{r_n}{r_1}\right)^{\sigma-1}\left|\frac{\cos[(s-1)(\pi-\theta_n)]}{\cos[(s-1)(\pi-\theta_1)]}\right| \le \sum_{n=2}^{\infty}\left(\frac{r_n}{r_1}\right)^{\sigma-1}$$

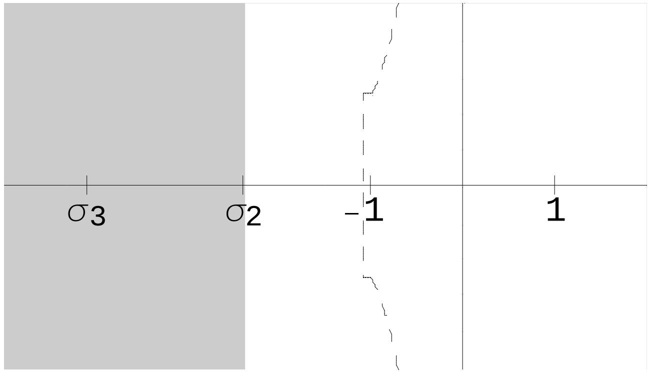
$$\le \phi(\sigma)$$

$$< 1$$

This establishes our Main Theorem.

Open Problems

1. How far to the right can the zero-free left half-plane $\{\mathrm{Re}(s) < \sigma_2\}$ be extended for $\zeta_2(s)$?



2. Does $\zeta_2(s)$ have any zero-free regions for $\mathrm{Re}(s) > 1$?

3. Can zero-free regions be established for every $\zeta_N(s)$ using similar techniques?

References

1. Davenport, H. and Heilbronn, H., *On the zeros of certain Dirichlet series*, J. London Math. Soc. 111 (1936), 181-185.
2. Hassen, A. and Nguyen, H. D., *Hypergeometric Zeta Functions*, Preprint, 2005.
3. Howard, F. T., *Numbers Generated by the Reciprocal of* $e^x - 1 - x$, Mathematics of Computation **31** (1977), No. 138, 581-598.
4. Spira, R., *Zeros of Hurwitz Zeta Function*, Mathematics of Computation 30 (1976), No. 136, 863-866.