The Story of Wavelets[1]

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Abstract: The theory and applications of wavelets have undoubtedly dominated the journals in all mathematical, engineering and related fields throughout the last decade. Few other theoretical developments in mathematical sciences have enjoyed this much attention and popularity, have been applied to such a diverse field of disciplines, and perhaps, have been so blindly misused. What was the missing piece in the great puzzle of signal processing, and how did the wavelets fill-in this missing piece? How did it all start, what development stages did it go through and what is the state of the art today? Have we reached the saturation, or do we have a long way to go? In this paper, we present three overviews in an attempt to answer these questions. In a historical overview, we look at the genesis of the wavelet theory as we take a short chronological journey through the wavelet times. In a technical overview, we look at the driving forces that played a key role in the development of the theory of wavelets, and try to find out what was so special that brought them to the center stage of scientific journals. In an application overview, we look at some of the most creative conventional and non-conventional applications of wavelets. On the conventional front, we discuss such applications as image compression, speech processing, and solution of partial differential equations. On the unconventional front, we look at various fields of applications including chemistry, neurophysiology, nondestructive evaluation, fractals, and economics. In particular, we discuss analyzing brain signals for the detection of Alzheimer's disease, analyzing ultrasonic weld inspection signals for the detection of cracks in piping of nuclear power plants, and analyzing fluctuations of financial markets. Finally, we look at new, noteworthy and promising developments, such as wavelet networks, and zero crossing representations, as we conclude what appears to be one of the most remarkable success stories of mathematical and engineering sciences: the story of wavelets. © Appeared in IMACS/IEEE CSCC'99 Proceedings, Pages:5481-5486

Key Words: Wavelet transforms, multiresolution analysis, historical and application overview

1 Introduction

A mathematician comes up with a good idea, develops a concrete theory, faces great opposition from other prominent figures in the area, but continues to work nevertheless. Then come in engineers and physicists, reformulate and modify that theory to make it more accessible, and eventually that idea becomes a standard tool for many researchers in many fields. Does this story sound familiar? The history of mathematics and engineering is full of such stories, but the similarity between two particular ones is quite striking.

In 1807, a French mathematician, Joseph Fourier, discovered that all periodic functions could be expressed as a weighted sum of basic trigonometric functions. His ideas faced much criticism from Lagrange, Legendre and Laplace for lack of mathematical rigor and generality, and his papers were denied publication. It took Fourier over 15 years to convince them and publish his results. Over the next 150 years his ideas were expanded and generalized for non-periodic functions and discrete time sequences. The fast Fourier transform algorithm, devised by Cooley and Tukey in 1965 placed the crown on Fourier transform, making it the king of all transforms. Since then Fourier transforms have been the most widely used, and often misused, mathematical tool in not only electrical engineering, but in many disciplines requiring function analysis.

[1] Invited plenary talk for special session on Wavelets and Nonlinear Processes in Physics

This crown however, is about to change hands. Following a remarkably similar history of development, the wavelet transform is rapidly gaining popularity and recognition. With applications ranging from pure mathematics to virtually every field of engineering, from astrology to economics, from oceanography to seismology, wavelet transforms are being applied to such areas where no other transform has ever been applied.

This paper is about the success story of wavelet transforms, and we look at this story from three different perspectives. From a historical perspective, we trace the development of wavelet theory from its early stages to where it is today in a chronological journey. From a technical perspective, we review the driving forces behind the wavelet theory, and compare wavelets to some of the other techniques that attempt to solve the same problem as wavelet transforms. Our aim in doing so is to show some of the reasons that brought this unparalleled fame and attention to wavelet transforms. Finally, from an application perspective, we summarize many conventional and non-conventional applications, and hope to form a bridge between researchers using the same tool for seemingly unrelated applications.

In recognition of the audience, the purpose of this paper is not to give a rigorous technical tutorial in wavelets, but rather a very non-formal overview of the field. Furthermore, it is not even intended to be complete or exhaustive. Considering that this overview is written by a *freshman researcher*, who was still in grade school by the time the modern theory of wavelets were laid out, such a goal would be too ambitious, and considering the space restrictions, such a goal would be impossible to achieve. Consequently, and due to tone of this paper, references are mostly omitted, except for certain applications mentioned in the last section, and a few general ones for the historical development of wavelets. With apologies for omitting many of the prominent names who made wavelets possible today, my goal in writing this paper is therefore two folds: to present a rather personal and perhaps a -behind the scenes- historical overview, and to point out to many different areas of applications of wavelets in an attempt to be a source of inspiration for new developments and applications.

2 A Historical Overview of Wavelets[1, 2]

The original idea belongs to Fourier: Approximate a complex function as a weighted sum of simpler functions, which themselves are obtained from one simple prototype function. The prototype function, also known as basis function, can then be though of as a building block, and the original function can be approximated, or under certain conditions be fully represented, by using similar building blocks. There are many advantages to such approximations and representations, as they provide valuable insight to analysis of complicated functions. Furthermore, if only a few of these building blocks renders good approximation, then significant compression can be obtained for the representation of the original function. Fourier used sinusoids of varying frequencies as building blocks and this representation provided us with the frequency content of the original function / signal. Fourier representations have been used in a variety of fields that called for signal analysis. However, these representations had one major drawback due to using sinusoids as basis functions. Sinusoids have perfect compact support in frequency domain, but not in time domain. In other words, they stretch out to infinity in time, and therefore, they cannot be used to approximate non-stationary signals. Note that the time domain representation of a signal does not provide any quantitative information about the spectral content of the signal. On the other hand, the Fourier representation only provides such spectral content with no indication about the time localization of the spectral components. Therefore, the analysis of non-stationary signals, whose spectral content change in time, requires a time-frequency representation (TFR), rather than just a frequency representation.

The first modification to the Fourier transform to allow analysis of non-stationary signals came as the short time Fourier transform (STFT). The idea behind the STFT was segmenting the signal by using a time-localized window, and performing the analysis for each segment. Since the Fourier transform was computed for every windowed (that is, time-localized) segment of the signal, STFT was able to provide a true time-frequency representation. Dennis Gabor, who was interested in representing a communication signal using oscillatory basis functions in a time frequency plane, was the first one to modify the Fourier transform into STFT in 1946. Shortly after, in 1947, Jean Ville devised a similar TFR for representing the energy of a signal in the time-frequency plane (the Wigner-Ville transform). Many other TFRs have been developed between late 1940s and early 1970s, each of which differed from the other ones only by the selection of the windowing function.

However, all these TFRs suffered from one major drawback: they all used the same window for the analysis of the entire signal. In late 1970s, J. Morlet, a geophysical engineer, was faced with the problem of analyzing signals which had very high frequency components with short time spans, and low frequency components with long time spans. STFT was able to analyze either high frequency components using narrow windows (wideband frequency analysis), or low frequency components using wide windows (narrowband frequency analysis), but not both. He therefore came up with the ingenious idea of using a different window function for analyzing different frequency bands. Furthermore, these windows were all generated by dilation or compression of a prototype Gaussian. These window functions had compact support both in time and in frequency (since the Fourier transform of a Gaussian is also a Gaussian). Due to the "small and oscillatory" nature of these window functions, Morlet named his basis functions as *wavelets of constant shape*. Just like Fourier, however, Morlet also faced much criticism from his colleagues. In 1980, looking for help to find a mathematically rigorous basis to his approach, Morlet met A. Grossman, a theoretical physicist of quantum mechanics who helped him to formalize the transformation and devised the inverse transformation. Little did they know, however, that the wavelet transform they developed was merely a rediscovery, and perhaps a slightly different interpretation of Alberto Calderón's 1964 work on harmonic analysis.

Yves Meyer, a French mathematician, who noticed the similarity between Morlet's and Calderón's work in 1984, also noticed that there was a great deal of redundancy in Morlet's choice of basis functions (which were then known as wavelets). Fascinated by this elegant nonstationary function analysis scheme, Meyer started working on developing wavelets with better localization properties. In 1985, he constructed orthogonal wavelet basis functions with very good time and frequency localization. Quite ironically, however, it turned out that another harmonic analyst, J.O. Strömberg had already discovered the very same wavelets about five years ago. Also it should be added that neither Meyer, nor Strömberg were the first two discover orthonormal wavelet basis functions. That honor goes way back to 1909, to a German mathematician, Alfred Haar. Although Haar wavelets are the first and the simplest orthonormal wavelets, they are of little practical use due to their poor frequency localization. Again, as a twist of history, it was later discovered that Haar's work on developing orthonormal basis functions were expanded in 1930s by Paul Levey, who was studying random signals of Brownian motion, and independently by Littlewood and Paley, who were working on localizing the contributing energies of a function.

In the mean time, Ingrid Daubechies, a former graduate student of Grossman at the Free University of Brussels, developed the wavelet frames for discretization of time and scale parameters of the wavelet transform, which allowed more liberty in the choice of basis functions at an expense of some redundancy. Daubechies, along with Stephane Mallat, is therefore credited with developing the transition from continuous to discrete signal analysis. In particular, in 1986 Mallat, a graduate student at Upenn, developed the idea of multiresolution analysis (MRA) for discrete wavelet transform (DWT) with Meyer, which later became his Ph.D. dissertation in 1988. The idea was decomposing a discrete signal into its dyadic frequency bands by a series of lowpass and highpass filters to compute its DWT from the approximations at these various scales. This idea, on the other hand, was all too familiar to electrical engineers for about twenty years under the name of quadrature mirror filters (QMF) and subband filtering, which were developed by A. Croisier, D. Esteban and C. Galand around 1976. Mallat's work constituted a natural extension of time localization to the well-established frequency localization idea of QMF and subband coding. Also in 1988, with the development of Daubechies' orthonormal bases of compactly supported wavelets, the foundations of the *modern* wavelet theory were laid.

The last ten years mostly have witnessed a search for other wavelet basis functions with different properties and modifications of the MRA algorithms. In 1992, Albert Cohen, Jean Feauveau and Daubechies constructed the compactly supported biorthogonal wavelets, which are preferred by many researchers over the orthonormal basis functions, whereas R. Coifman, Meyer and Victor Wickerhauser developed wavelet packets, a natural extension of MRA.

3 A Technical Overview of Wavelets[3]

The earliest form of function representation using orthogonal basis functions is undoubtedly the Fourier series for continuous and periodic signals:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk(2\pi/T)t}, \; c_k = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt \quad (1)$$

where $x(t)$ is the signal to be analyzed, T is the period of the signal, and c_k are the Fourier coefficients, representing the spectral components of $x(t)$. The complex exponential functions at different discrete frequencies of $2\pi jk/T$ are not compactly supported in time since they extend to infinity (though they are perfectly compactly supported in frequency since the Fourier transform of an exponential at frequency $2\pi jk/T$ is a delta function at this frequency). As we noted above, this makes the Fourier representation inadequate in analyzing non-stationary signals. In other words, due their infinite time support, complex exponentials analyze the signal globally in time, and can only tell what spectral components exist in the signal. Fourier representation cannot provide any information regarding the time localization of these spectral components. This is not a problem for analyzing stationary signals, since all spectral components exist at all times. For non-stationary signals, however, whose spectral content change in time, Fourier representation is clearly not appropriate. Unfortunately, most signals encountered in practice, regardless of their source, are non-stationary in nature. Many people who did not realize this shortcoming, blindly (mis)used Fourier representation for analyzing non-stationary signals.

The STFT was a much needed modification which allowed analysis of non-stationary signals by segmenting them into -stationary enough- short pieces, and computing the Fourier representation of each piece:

$$\begin{aligned} S(\tau, f) &= \int x(t) w^*(t-\tau) e^{-2j\pi ft} dt \\ x(t) &= \int_\tau \int_f S(\tau, f) w^*(t-\tau) e^{2j\pi ft} d\tau df \end{aligned} \tag{2}$$

where $w(t)$ is the windowing function, f and τ are frequency and translation (time) parameters respectively, * is the complex conjugate operator, and $S(\tau,f)$ is the STFT of $x(t)$ at frequency f and translation τ. Note that for each frequency f, time localization is obtained through segmenting $x(t)$ by $w(t-\tau)$, the windowing function centered at $t=\tau$. The Fourier transform of this segmented signal then provides the frequency localization, which is what Fourier transform does best.

The problem with this approach is that it provides constant resolution for all frequencies since it uses the same window for the analysis of the entire signal. If the signal to be analyzed has high frequency components for a short time span, a narrow window (compactly supported in time) would be necessary for good time resolution. Note, however, that narrow windows mean wider frequency bands, resulting in poor frequency resolution. If, on the other hand, the signal also features low frequency components of longer time span, than a wider window need to be used to obtain good frequency resolution (at the expense of time resolution).

This was precisely the driving force behind the wavelet transform (WT), which provides varying time and frequency resolutions by using windows of different lengths. In essence, WT actually does the opposite of STFT by first decomposing the signal into frequency bands, and then analyzing them in time:

$$\begin{aligned} W(a,b) &= \frac{1}{\sqrt{a}} \int x(t) \psi^*\left(\frac{t-b}{a}\right) dt \\ x(t) &= \frac{C_\psi}{a^2} \int_{a>0} \int_b W(a,b) \psi^*\left(\frac{t-b}{a}\right) da \cdot db \end{aligned} \tag{3}$$

where $a>0$ and b are scale and translation parameters, respectively, ψ is the mother wavelet, C_ψ is a constant that depends on ψ, and $W(a,b)$ is the continuous wavelet transform of $x(t)$. Note that we can interpret Equation 3 as an inner product of $x(t)$ with the scaled and translated versions of the basis functions ψ:

$$\begin{aligned} & W(a,b) = \int x(t) \psi^*{}_{(a,b)}(t) dt, \text{ where} \\ & \psi_{(a,b)}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right), \quad a>0, b \in \Re \end{aligned} \tag{4}$$

Note that scaled and translated versions of the basis functions are obtained from one prototype function, the mother wavelet. It is also worth mentioning that the name wavelet originates from the admissibility condition, which requires the basis functions to be of finite support (small) and of oscillatory (wavy) nature, hence wavelet (small wave).

To obtain the DWT, the papramenters a and b need to discretized. Daubechies showed that discretizing by $a=2^j$ and $b=2^jk$ will yield orthonormal basis functions for certain choices of ψ (Daubechies wavelets)

$$\psi_{(j,k)}(t) = 2^{-j/2} \psi(2^{-j} t - k) \tag{5}$$

Mallat showed that MRA can then be used to obtain the DWT of a discrete signal by iteratively applying lowpass and highpass filters, and subsequently down sampling them by two. Figure 1 shows this procedure, where g[n] and h[n] are the highpass and lowpass filters, respectively. Also shown in the figure are frequency bands (in terms discrete frequencies) for each level. At each level, this procedure computes

$$\begin{aligned} y_{high}[k] &= \sum_n x[n] \cdot g[2k-n] \\ y_{low}[k] &= \sum_n x[n] \cdot h[2k-n] \end{aligned} \tag{6}$$

where

$$h[N-1-n] = (-1)^n g[n] \tag{7}$$

with N being the total number of samples in $x[n]$.

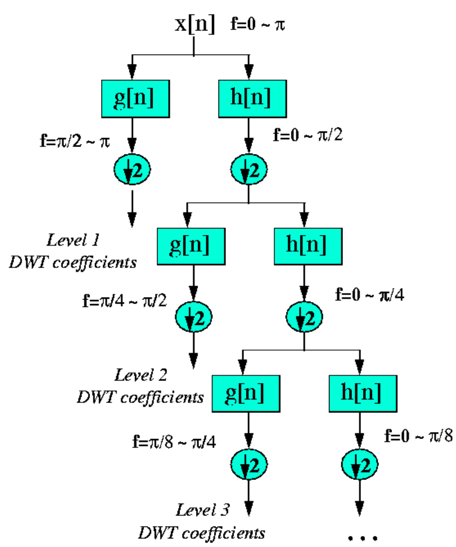

Figure 1. Computing DWT by MRA

The original signal can be reconstructed by following the exact opposite steps, or by computing

$$x[n] = \sum_k \left(y_{high}[k] \cdot g[2k-n]\right) + \left(y_{low}[k] \cdot h[2k-n]\right) \quad (8)$$

where y_{high} and y_{low} are the outputs of highpass and lowpass filters, respectively, at each level.

So what made the wavelet transform so popular? There are a number of reasons. Wavelet transforms provide what many researchers needed for a very long time: A systematic approach for analyzing non-stationary signals. Although various other TFRs existed for over five decades, they had their own limitations. For example, many, including STFT, are not able to analyze signals with both sharp transitions and slowly varying spectra. This is because these TFRs are based on computing windowed Fourier transforms using a constant window. Many other TFRs, on the other hand, are quadratic or non-linear in nature with computational difficulties. Wavelet transform is the only linear transform that can analyze non-stationary signals at varying resolutions by decomposing the signals into their frequency bands. Furthermore, DWT is a very fast algorithm with polynomial time and space complexity, which makes it even more appealing.

4 An Application Overview of Wavelets

The application areas for wavelets have been growing for the last ten years at a very rapid rate. Reviewing all of them in a couple of pages is certainly not possible. The purpose of this section is to point out to various areas that wavelets can be used, and hopefully be a source of inspiration for new research.

Data Compression[4]: Apart from its original intention of analyzing non-stationary signals, wavelets have been most successful in image processing and compression applications. Subband coding have long been used for compression, so using DWT has been a natural extension. Due to the compact support of the basis functions used in wavelet analysis, wavelets have good energy concentration properties. Most DWT coefficients usually are therefore very small, and they can be discarded without incorporating a significant error in the reconstruction stage.

Denoising: Compression property has been further explored by Iain Johnstone and David Donoho[5] for denoising applications, and they have devised the *wavelet shrinkage denoising (WSD).* The idea behind WSD is based on recognizing that noise will show itself at finer scales, and discarding the coefficients that fall below a certain threshold at these scales will remove the noise.

Source and Channel Coding[6]: Wavelets fit naturally into source coding and channel coding problems, since source coding requires developing a very compact representation of the information to be transmitted, and channel coding requires incorporating controlled amounts of redundancy into the representation to reduce the ill-effects of channel noise.

Biomedical Engineering: Due to the very nature of all biological signals being non-stationary, wavelets have also enjoyed great success in biomedical engineering. Wavelets have been used for the analysis of electrocardiogram for diagnosing cardiovascular disorders, and of electroencephalogram for diagnosing neurophysiological disorders, such as seizure detection, or analysis of evoked potentials for detection of Alzheimer's disease[7]. Wavelets have also been used for the detection of microcalcifications in mammograms and processing of computer tomography and magnetic resonance images.

Nondestructive Evaluation: Another interesting area of applications has been nondestructive evaluation (NDE). Wavelets have been successfully used for the analysis of ultrasonic and eddy current NDE signals for flaw detection in various media such as nuclear power plant tubings[8], gas pipelines, aircraft components, etc.

Numerical Solution of PDEs: Partial differential equations have been successfully discretized by using wavelets as basis functions, and then solved numerically. This also gave rise to new methods in finite element analysis.

Study of Distant Universes[9]**:** One of the more unconventional applications of wavelets has been on hierarchical organization of distant galaxies. Recognizing that distribution of galaxies forms hierarchical structures at various scales, Albert Bijaoui developed a multi-scale vision model using wavelets for classifying each component in this hierarchy.

Wavelet Networks: The success of radial basis function (RBF) neural networks for function approximation was a good indicator of this yet another field of application of wavelets. The excellent time and frequency localization properties of wavelets as basis functions replaced Gaussian functions of RBF networks. In their pioneering 1993 paper, Bakshi and Stephanopoulos[10] showed that neural networks using wavelets as basis functions are particularly efficient in learning from sparse data, since using a higher resolution of the space when the data is dense, and a lower resolution when data is sparse also fit naturally into multiresolution wavelet analysis scheme. More recently, Bernard, Mallat and Slotine proposed wavelet interpolation networks capable of real time learning of unknown functions.

Zero Crossing Representation: Discovered by David Marr in early 1980s, and developed by Mallat in late 1980s, zero crossing representations of wavelet coefficients have also found significant applications in signal classification, computer vision, data compression and signal denoising[2]. More recently M. Afzal devised a new approach for shift invariant, unique and complete representation of signals using zero crossings of wavelet based multi resolution decompositions of time domain signals.

Fractals: Certain wavelets, such as the Daubechies wavelets have a fractal (self-similar) structure, and when combined with multiresolution formulation, they provide a very natural way of analyzing fractals. Marie Farge, G. Wornell and Alan Oppenheim have successfully applied wavelets to fractal analysis[4].

Turbulence Analysis[2]**:** Again due to the multiresolution analysis properties of wavelets, there has been significant effort applying them to analysis of turbulent flow of low viscosity fluids flowing at high speeds. In particular, efforts are underway to solve Navier-Stokes equations using wavelet based numerical techniques by Marie Farge and Gregory Beylkin

Financial Analysis: Financial and economic data, such as stock prices, are usually analyzed in either time or frequency domain. Since these data are often in the form of highly non-stationary time series, simultaneously analyzing time and frequency dependencies using multiresolution wavelet analysis provide valuable insight to financial communities.

5 Conclusions

Wavelet analysis has enjoyed a tremendous attention and success over the last decade, and for a good reason. Almost all signals encountered in practice call for a time-frequency analysis, and wavelets provide a very simple and efficient means of performing such an analysis. So what is next? The theoretical developments of wavelets have been largely completed over the last two decades (or may be we should say over the last 90 years!). The previous list of applications of wavelets is by no means complete or exhaustive, and considering the endless variety of non-stationary signals that are commonly encountered in engineering, mathematical and natural sciences, it is not difficult to estimate that there will be significant search for new fields of applications.

References:

[1] I. Daubechies, Where Do Wavelets Come From?, *Proceedings of the IEEE* , Vol. 84, No. 4, 1996, pp. 510-513.

[2] Y. Meyer, *Wavelets, Applications and Algorithms*, Siam, 1993

[3] O. Rioul and M. Vetterli, Wavelets and Signal Processing, *IEEE SP Magazine*, Vol. 8, No. 10, 1991, pp.14-38.

[4] R. Rao and A. Bopardikar, *Wavelet Transforms, Introduction to Theory and Applications*, Addison Wesley, 1998.

[5] D. Donoho, Denoising via Soft Thresholding, *IEEE Tran. on Information Theory*, Vol. 41, No. 5, 1995, pp.613-627.

[6] G. Wornell, Emerging Applications of Multirate Signal Processing and Wavelets in Digital Communications, *Proceedings of the IEEE*, Vol. 84, No. 4, 1996, pp. 586-603.

[7] R. Polikar, M.H. Greer, L. Udpa, F. Keinert, Multiresolution Wavelet Analysis of ERPs for the Detection of Alzheimer's Disease, Proceedings, *19th Intl. Conf. IEEE/EMBS, 1997, pp.1301-1304.*

[8] R. Polikar, L. Udpa, S.S. Udpa, T. Taylor, Frequency Invariant Classification of Weld Inspection Signals, *IEEE Tran. Ultra. Ferro and Freq. Cont.* Vol. 45, No. 3, 1998, pp.614-625.

[9] A. Bijaoui, E. Slezak, F. Rue, and E. Lega, Wavelets and the Study of the Distant Universe, *Proceedings of the IEEE*, Vol. 84, No. 4, 1996, pp. 670-679.

[10] B. Bakshi, G. Stephanopoulos, Wave-Net: A Multiresolution, Hierarchical Neural Network with Localized learning, *AIChE Journal*, Vol. 39, No. 1, 1993, pp.57-81.