Homework #6

Due 11:59pm (Section 2/Section 3) March 29, 2019

You are encouraged to read Chapters 2.6 of your textbook to complete the homework

Please follow the name convention to name your **folders and files**: *CC_<lastName>_HW<#>.pdf*. **Your HW will not be graded if you do not follow this name convention. And** electronically submit your HW to cv094021@gmail.com

Pages 155-159

4. (a) The marginal probability mass function $p_X(x)$ is found by summing along the rows of the joint probability mass function.

x	y = 0	1	2	3	$p_X(x)$
0	0.15	0.12	0.11	0.10	0.48
1	0.09	0.07	0.05	0.04	0.25
2	0.06	0.05	0.04	0.02	0.17
3	0.04	0.03	0.02	0.01	0.10
$p_Y(y)$	0.34	0.27	0.22	0.17	

$p_X(0) = 0.48$, $p_X(1) = 0.25$, $p_X(2) = 0.17$, $p_X(3) = 0.10$, $p_X(x) = 0$ if $x \neq 0, 1, 2,$ or 3

(b) The marginal probability mass function $p_Y(y)$ is found by summing down the columns of the joint probability mass function. So $p_Y(0) = 0.34$, $p_Y(1) = 0.27$, $p_Y(2) = 0.22$, $p_Y(3) = 0.17$, $p_Y(y) = 0$ if $y \neq 0, 1, 2,$ or 3

(c) No, X and Y are not independent. For example, $P(X = 0 \text{ and } Y = 0) = 0.15$, but $P(X = 0)P(Y = 0) = (0.48)(0.34) = 0.1632 \neq 0.15$.

(d) $\mu_X = 0p_X(0) + 1p_X(1) + 2p_X(2) + 3p_X(3) = 0(0.48) + 1(0.25) + 2(0.17) + 3(0.10) = 0.89$

$\mu_Y = 0p_Y(0) + 1p_Y(1) + 2p_Y(2) + 3p_Y(3) = 0(0.34) + 1(0.27) + 2(0.22) + 3(0.17) = 1.22$

(e) $\sigma_X^2 = 0^2p_X(0)+1^2p_X(1)+2^2p_X(2)+3^2p_X(3)-\mu_X^2 = 0^2(0.48)+1^2(0.25)+2^2(0.17)+3^2(0.10)-0.89^2 = 1.0379$

$\sigma_X = \sqrt{1.0379} = 1.0188$

$\sigma_Y^2 = 0^2p_Y(0)+1^2p_Y(1)+2^2p_Y(2)+3^2p_Y(3)-\mu_Y^2 = 0^2(0.34)+1^2(0.27)+2^2(0.22)+3^2(0.17)-1.22^2 = 1.1916$

$\sigma_Y = \sqrt{1.1916} = 1.0916$

(f) $\text{Cov}(X,Y) = \mu_{XY} - \mu_X\mu_Y$

$$\begin{aligned}
\mu_{XY} &= (0)(0)p_{X,Y}(0,0) + (0)(1)p_{X,Y}(0,1) + (0)(2)p_{X,Y}(0,2) + (0)(3)p_{X,Y}(0,3) + (1)(0)p_{X,Y}(1,0) \\
&\quad + (1)(1)p_{X,Y}(1,1) + (1)(2)p_{X,Y}(1,2) + (1)(3)p_{X,Y}(1,2) + (2)(0)p_{X,Y}(2,0) + (2)(1)p_{X,Y}(2,1) \\
&\quad + (2)(2)p_{X,Y}(2,2) + (2)(3)p_{X,Y}(2,2) + (3)(0)p_{X,Y}(3,0) + (3)(1)p_{X,Y}(3,1) + (3)(2)p_{X,Y}(3,3) \\
&\quad + (3)(3)p_{X,Y}(3,3) \\
&= (0)(0)(0.15) + (0)(1)(0.12) + (0)(2)(0.11) + (0)(3)(0.10) \\
&+ (1)(0)(0.09) + (1)(1)(0.07) + (1)(2)(0.05) + (1)(3)(0.04) \\
&+ (2)(0)(0.06) + (2)(1)(0.05) + (2)(2)(0.04) + (2)(3)(0.02) \\
&+ (3)(0)(0.04) + (3)(1)(0.03) + (3)(2)(0.02) + (4)(3)(0.01) \\
&= 0.97
\end{aligned}$$

$\mu_X = 0.89$, $\mu_Y = 1.22$

$\text{Cov}(X,Y) = 0.97 - (0.89)(1.22) = -0.1158$

14. (a) $T = 50X + 100Y$, so $\mu_T = \mu_{50X+100Y} = 50\mu_X + 100\mu_Y = 50(1.01) + 100(1.23) = 173.50$.

(b)
$$\begin{aligned}
\sigma_T &= \sigma_{50X+100Y} \\
&= \sqrt{50^2\sigma_X^2 + 100^2\sigma_Y^2 + 2(50)(100)\text{Cov}(X,Y)} \\
&= \sqrt{50^2(0.8099) + 100^2(0.9971) + 2(50)(100)(0.2377)} \\
&= 119.9
\end{aligned}$$

(c) $P(T = 250) = P(X = 1 \text{ and } Y = 2) + P(X = 3 \text{ and } Y = 1) = 0.08 + 0.02 = 0.10$

15. (a) $p_{Y|X}(0\,|\,3) = \dfrac{p_{X,Y}(3,0)}{p_X(3)} = \dfrac{0.01}{0.07} = 0.1429$

$p_{Y|X}(1\,|\,3) = \dfrac{p_{X,Y}(3,1)}{p_X(3)} = \dfrac{0.02}{0.07} = 0.2858$

$p_{Y|X}(2\,|\,3) = \dfrac{p_{X,Y}(3,2)}{p_X(3)} = \dfrac{0.02}{0.07} = 0.2858$

$p_{Y|X}(3\,|\,3) = \dfrac{p_{X,Y}(3,3)}{p_X(3)} = \dfrac{0.02}{0.07} = 0.2858$

(b) $p_{X|Y}(0\,|\,1) = \dfrac{p_{X,Y}(0,1)}{p_Y(1)} = \dfrac{0.10}{0.34} = 0.2941$

$p_{X|Y}(1\,|\,1) = \dfrac{p_{X,Y}(1,1)}{p_Y(1)} = \dfrac{0.16}{0.34} = 0.4706$

$p_{X|Y}(2\,|\,1) = \dfrac{p_{X,Y}(2,1)}{p_Y(1)} = \dfrac{0.06}{0.34} = 0.1765$

$p_{X|Y}(3\,|\,1) = \dfrac{p_{X,Y}(3,1)}{p_Y(1)} = \dfrac{0.02}{0.34} = 0.0588$

(c) $E(Y\,|\,X=3) = 0p_{Y|X}(0\,|\,3) + 1p_{Y|X}(1\,|\,3) + 2p_{Y|X}(2\,|\,3) + 3p_{Y|X}(3\,|\,3) = 1.71.$

(d) $E(X\,|\,Y=1) = 0p_{X|Y}(0\,|\,1) + 1p_{X|Y}(1\,|\,1) + 2p_{X|Y}(2\,|\,1) + 3p_{X|Y}(3\,|\,1) = 1$

20. (a)
$$\begin{aligned} P(X>1 \text{ and } Y>1) &= \int_1^\infty \int_1^\infty 4xye^{-(2x+y)}\,dy\,dx \\ &= -\int_1^\infty \left(4x(1+y)e^{-(2x+y)} \Big|_1^\infty \right) dx \\ &= \int_1^\infty 8xe^{-(2x+1)}\,dx = -2(1+2x)e^{-(2x+1)} \Big|_1^\infty \\ &= 6e^{-3} \end{aligned}$$

(b) For $x \le 0$, $f_X(x) = 0$.

For $x > 0$, $f_X(x) = \displaystyle\int_0^\infty 4xye^{-(2x+y)}\,dy = -4x(1+y)e^{-(2x+y)} \Big|_0^\infty = 4xe^{-2x}.$

Therefore $f_X(x) = \begin{cases} 4xe^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$

For $y \le 0$, $f_Y(y) = 0$.

For $y > 0$, $f_Y(y) = \displaystyle\int_0^\infty 4xye^{-(2x+y)}\,dx = -y(1+2x)e^{-(2x+y)} \Big|_0^\infty = ye^{-y}.$

Therefore $f_Y(y) = \begin{cases} ye^{-y} & y > 0 \\ 0 & y \le 0 \end{cases}$

(c) Yes, $f(x,y) = f_X(x)f_Y(y)$.

Pages 160-161:

12. (a) A and B are mutually exclusive if $P(A \cap B) = 0$, or equivalently, if $P(A \cup B) = P(A) + P(B)$. So if $P(B) = P(A \cup B) - P(A) = 0.7 - 0.3 = 0.4$, then A and B are mutually exclusive.

    (b) A and B are independent if $P(A \cap B) = P(A)P(B)$. Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. So A and B are independent if $P(A \cup B) = P(A) + P(B) - P(A)P(B)$, that is, if $0.7 = 0.3 + P(B) - 0.3P(B)$. This equation is satisfied if $P(B) = 4/7$.

22. Let D denote the event that an item is defective, let S_1 denote the event that an item is produced on the first shift, let S_2 denote the event that an item is produced on the second shift, and let S_3 denote the event that an item is produced on the third shift. Then $P(S_1) = 0.50$, $P(S_2) = 0.30$, $P(S_3) = 0.20$, $P(D|S_1) = 0.01$, $P(D|S_2) = 0.02$, and $P(D|S_3) = 0.03$.

(a)
$$\begin{aligned} P(S_1|D) &= \frac{P(D|S_1)P(S_1)}{P(D|S_1)P(S_1) + P(D|S_2)P(S_2) + P(D|S_3)P(S_3)} \\ &= \frac{(0.01)(0.50)}{(0.01)(0.50) + (0.02)(0.30) + (0.03)(0.20)} \\ &= 0.294 \end{aligned}$$

(b)
$$\begin{aligned} P(S_3|D^c) &= \frac{P(D^c|S_3)P(S_3)}{P(D^c|S_1)P(S_1) + P(D^c|S_2)P(S_2) + P(D^c|S_3)P(S_3)} \\ &= \frac{[1 - P(D|S_3)]P(S_3)}{[1 - P(D|S_1)]P(S_1) + [1 - P(D|S_2)]P(S_2) + [1 - P(D|S_3)]P(S_3)} \\ &= \frac{(1 - 0.03)(0.20)}{(1 - 0.01)(0.50) + (1 - 0.02)(0.30) + (1 - 0.03)(0.20)} \\ &= 0.197 \end{aligned}$$