ECE 09468/09568
Discrete Event Systems

Lecture 10: **Petri Nets (PNs) – Part VI**

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Timed Petri Nets

- **Deterministic Timed PN (DTPN):**
  - Concurrent ***choice-free*** PN that associate ***fixed*** *time delays* to transitions, places, or arcs.
- **Stochastic Petri Nets (SPN):**
  - Time delays are associated with ***transition only***
  - Time delay for each transition is assumed to be random and ***exponentially distributed.***

What is Exponential Distribution

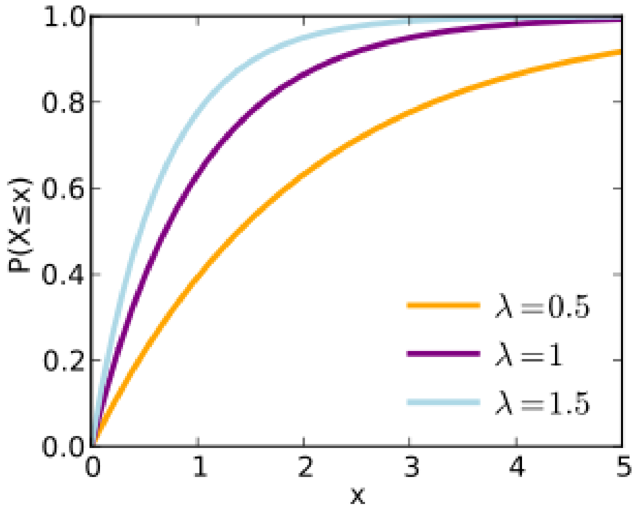
Given a random variable τ, if the probability that τ is greater than a given value x satisfies:

$$P(\tau>x)=e^{-\lambda x} \quad \text{for } x\geq 0$$

i.e. distribution function $F(x)$ = $\mathbf{P(\tau\leq x) = 1 - e^{-\lambda x}}$ Where $\lambda > 0$, called a rate.

τ is an **exponential random variable**.

If arrival intervals are such random variable, this arrival process is called **Poisson arrival process**.



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Exponential Random Variables

If τ is an exponential random variable, then

the mean (expected value) of τ is **$E(\tau)=1/\lambda$**

(Physically mean=average, λ=1/average is thus called **rate**)

A transition in a PN can be associated with an exponentially distributed random time delay. In this case, λ **is called firing rate.**

SPN Model

5-tuple : SPN =(P, T, I, O, Λ, m_0)

- P - A finite set non-empty of places
- T - A finite set non-empty of transitions
- I - A set of input arcs connecting P to T
- O - A set of output arcs connecting T to P
- $\Lambda: T \rightarrow R^+$ - Transition firing rates
- M_0 - Initial marking

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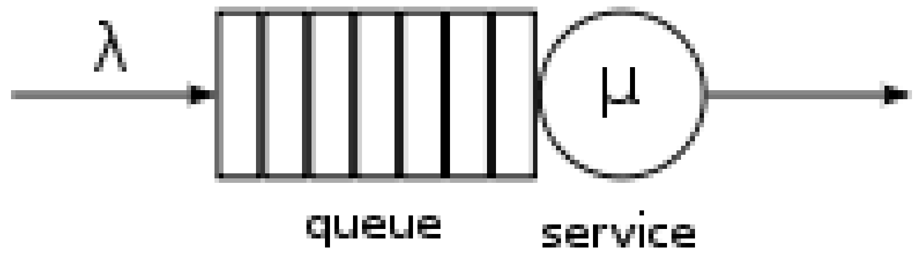
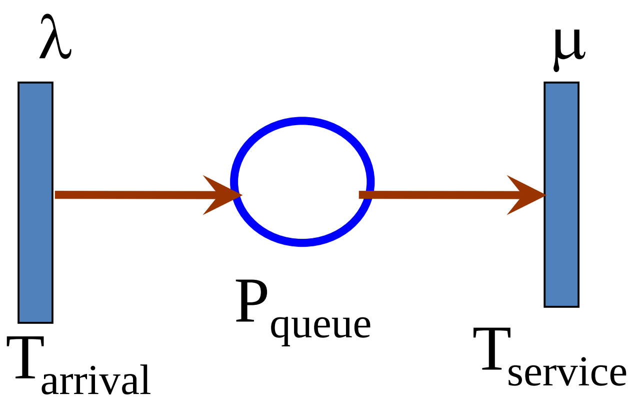
M/M/1 Queue – PN representation

The M/M/1 is a queue with exponential arrivals and service times and a single server.



PN representation



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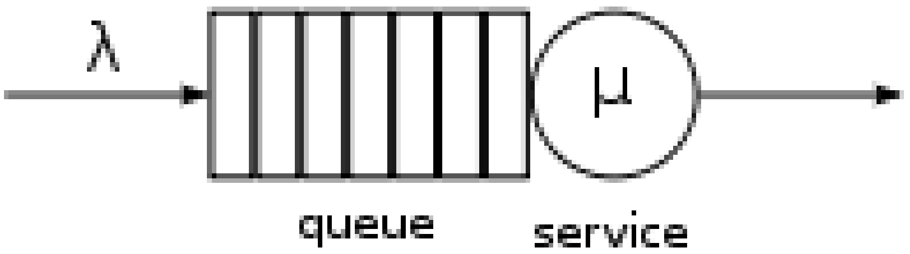
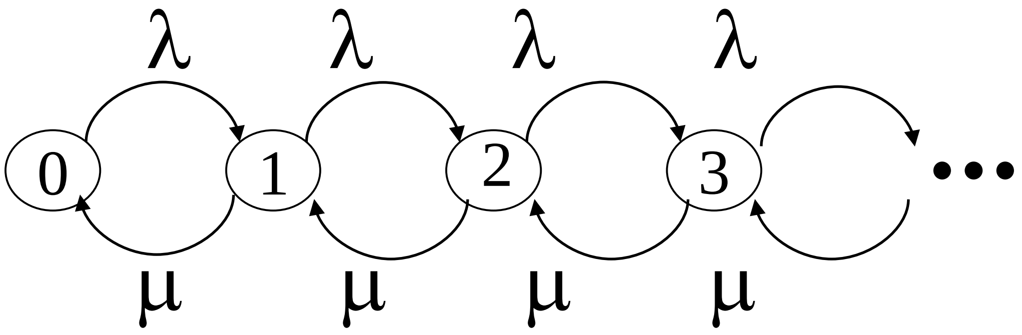
Markov Chain Model of M/M/1



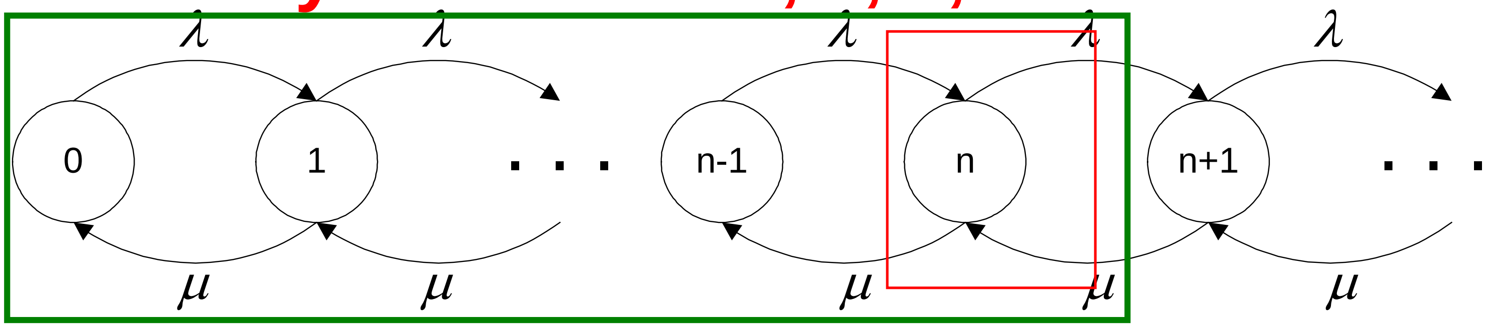
A **Markov chain** is a random process with the property that the next state depends only on the current state.



How to solve a Markov model for steady state probabilities, average wait time, and throughput?

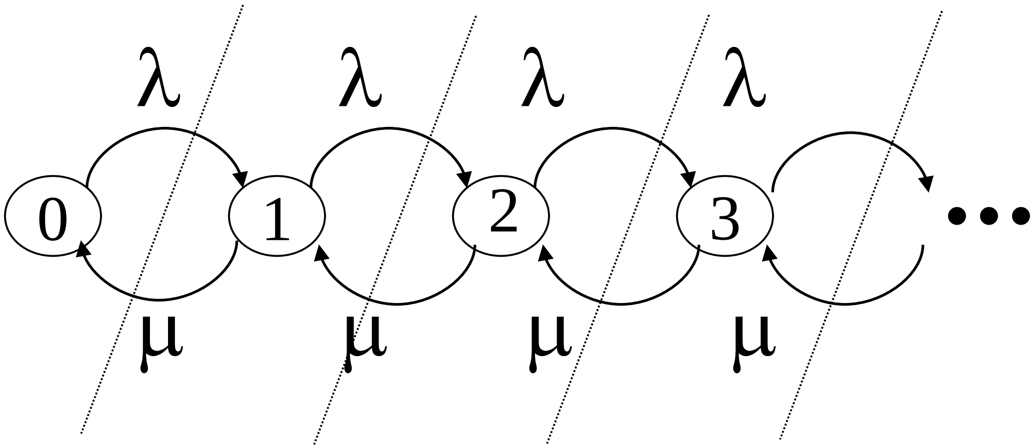
Markov Chain Model of M/M/1

- ***How to find the*** **π_i is the steady state probability for state i=0, 1, 2, …?**



- *Balance equation:* $(\lambda + \mu)\pi_n = \lambda\pi_{n-1} + \mu\pi_{n+1}$
- *Global balance equation:* $\lambda\pi_n = \mu\pi_{n+1}$
- *Normalization condition:* $\sum_{n=0}^{\infty}\pi_n = 1$

Solving Markov Model of M/M/1



List the balance equations (the probability flow from state n to state m equals the probability flow from state m to state n):

$$\mu\pi_1=\lambda\pi_0,\ \mu\pi_2=\lambda\pi_1,\ \mu\pi_3=\lambda\pi_2,\ldots$$

Solving Markov Model of M/M/1

Assume that π_0 is known and let $\rho=\lambda/\mu$, we find:

$$\mu\pi_1=\lambda\pi_0 \Rightarrow \pi_1= (\lambda/\mu)\pi_0= \rho\pi_0$$
$$\mu\pi_2=\lambda\pi_1 \Rightarrow \pi_2= (\lambda/\mu)\pi_1= \rho\pi_1= \rho^2\pi_0$$
$$\mu\pi_3=\lambda\pi_2 \Rightarrow \pi_3= (\lambda/\mu)\pi_2= \rho\pi_1= \rho^3\pi_0$$

…

Since the sum of all probabilities is one, we have:

$$\pi_0+\pi_1+\pi_2+\pi_3+ \ldots=1$$

$$\pi_0(1+\rho+\rho^2+\rho^3+ \ldots)=1$$

Solving Markov Model of M/M/1

1+ρ+ρ2+ρ3+ … is a geometric series

Stability condition for M/M/1: **ρ<1** (service faster than arrival). Then we have:

$$\pi_0=1-\rho$$

$$\pi_1= \rho(1-\rho)$$
$$\pi_2= \rho^2(1-\rho)$$
$$\pi_3= \rho^3(1-\rho)$$
…

Performance of M/M/1

If $\rho<1$ (service faster than arrival), the throughput is

$$\gamma=\mu\pi_1+\mu\pi_2+\mu\pi_3+\ldots$$
$$=\mu(\pi_1+\pi_2+\pi_3+\ldots)=\mu(1-\pi_0)=\mu\rho=\lambda$$

Average number of customers in the system:

$$E(N)=\pi_1+2\pi_2+3\pi_3+\ldots=\rho/(1-\rho)$$

Little's formula: $E(N)=\lambda E(T)$ where $E(T)$ is average delay

$$E(T)=E(N)/\lambda=[1/\mu]/(1-\rho) \text{ and}$$

Average wait: $E(W)=E(T)-1/\mu$

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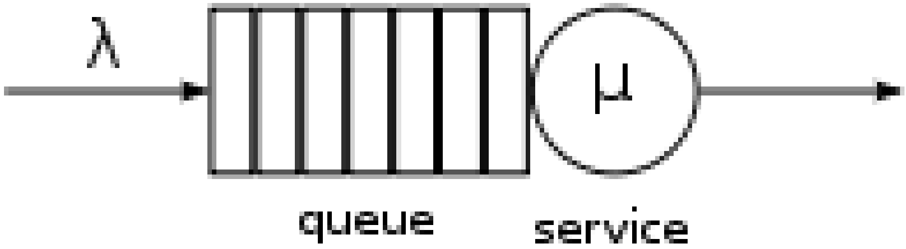
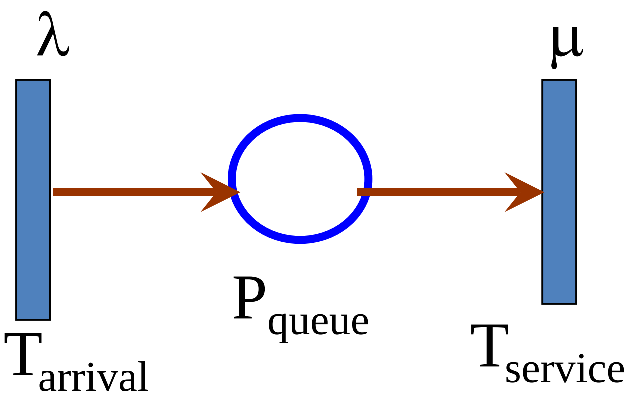
SPN (Stochastic Petri Net)

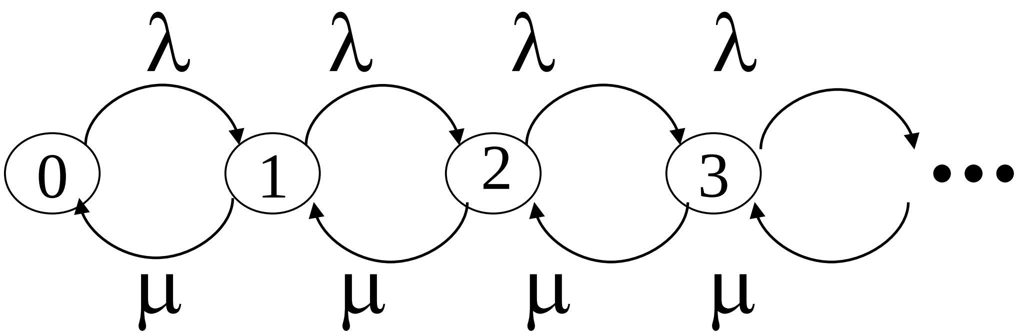
- SPNs are obtained by associating stochastic and timing information to Petri nets
  - Time delays are associated with *transition only*
  - Time delay for each transition is assumed to be random and *exponentially distributed.*

SPN model of M/M/1 Queue



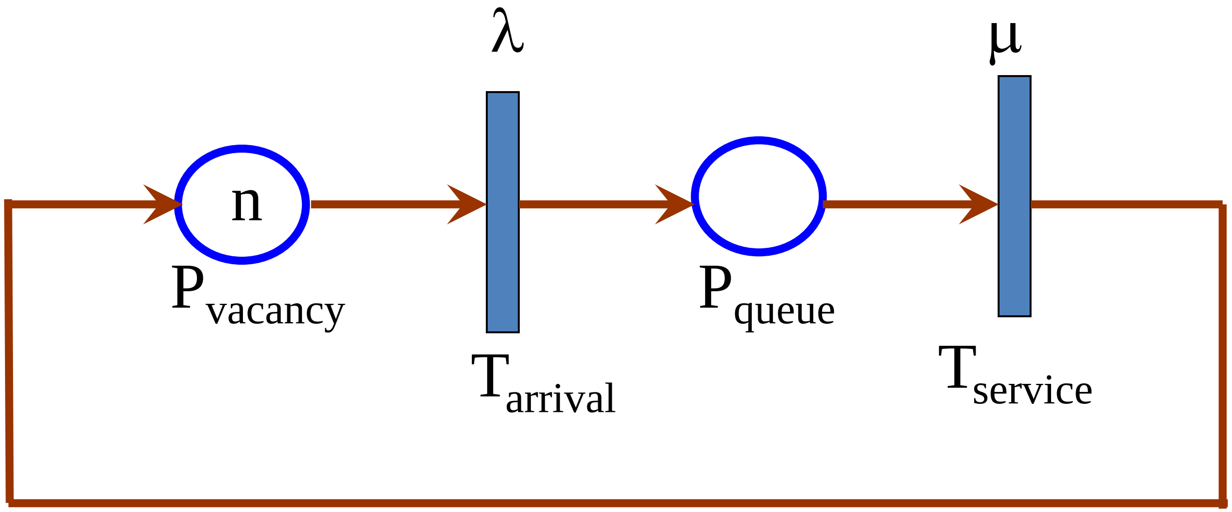
SPN representation





PN model of M/M/1/n Queue

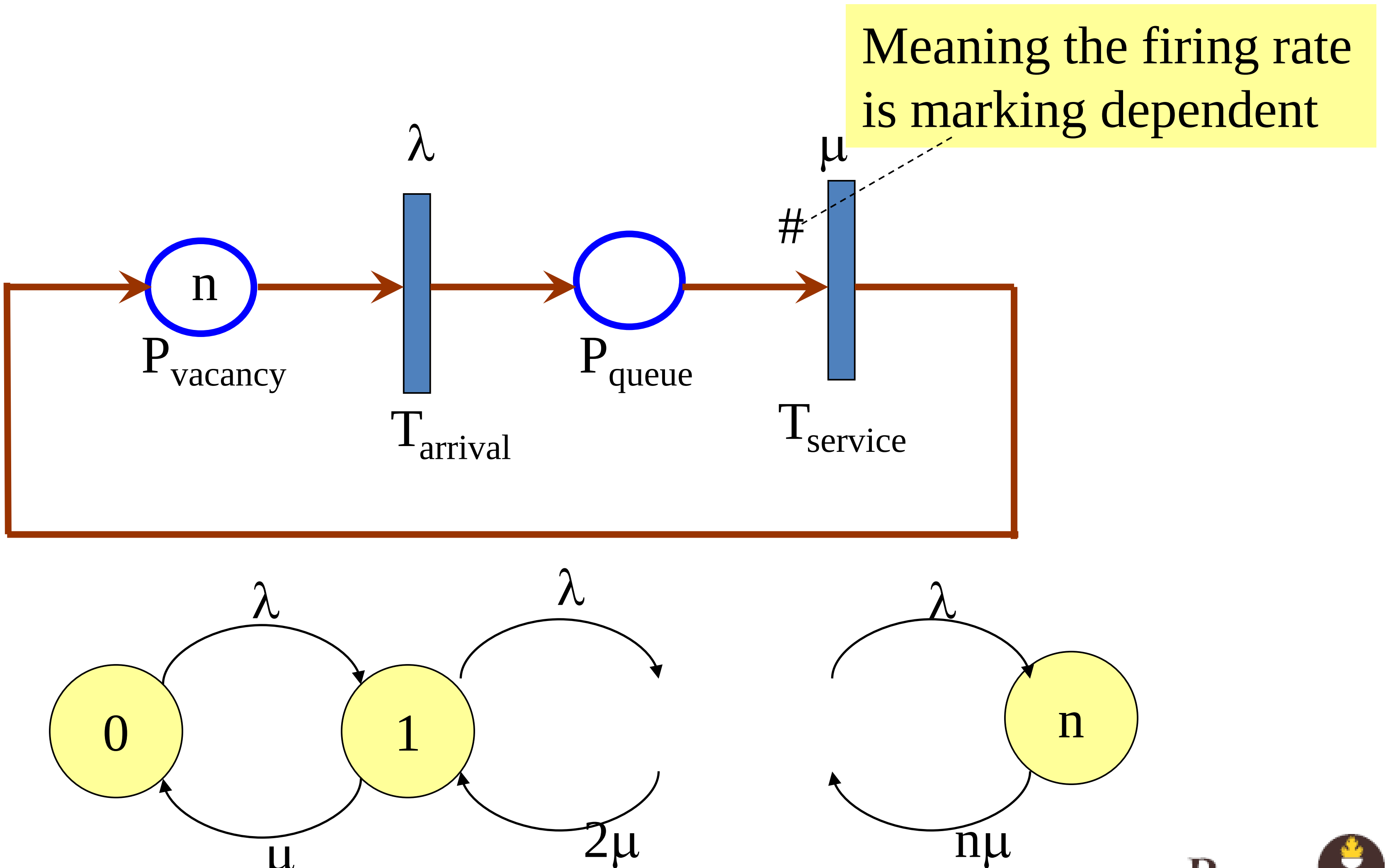
- M/M/1/n queue: an M/M/1 queuing system with a limited buffer space so that at most *n* jobs can be in the system at a time



What is the corresponding Markov model?

PN model of M/M/n/n Queue

- M/M/n/n queue: n servers and buffer size is n.



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Solution Procedure for SPN

- Build an SPN including proper time delay variables characterized by firing rates.
- Find reachability graph labeling with firing rates.
- Build up a transition rate matrix A and solve $\pi A=0$ and $\sum \pi_i=1$ where

$$\pi=(\pi_0, \pi_1, \ldots, \pi_{q-1})$$

Or list q-1 balance equations instead of $\pi A=0$

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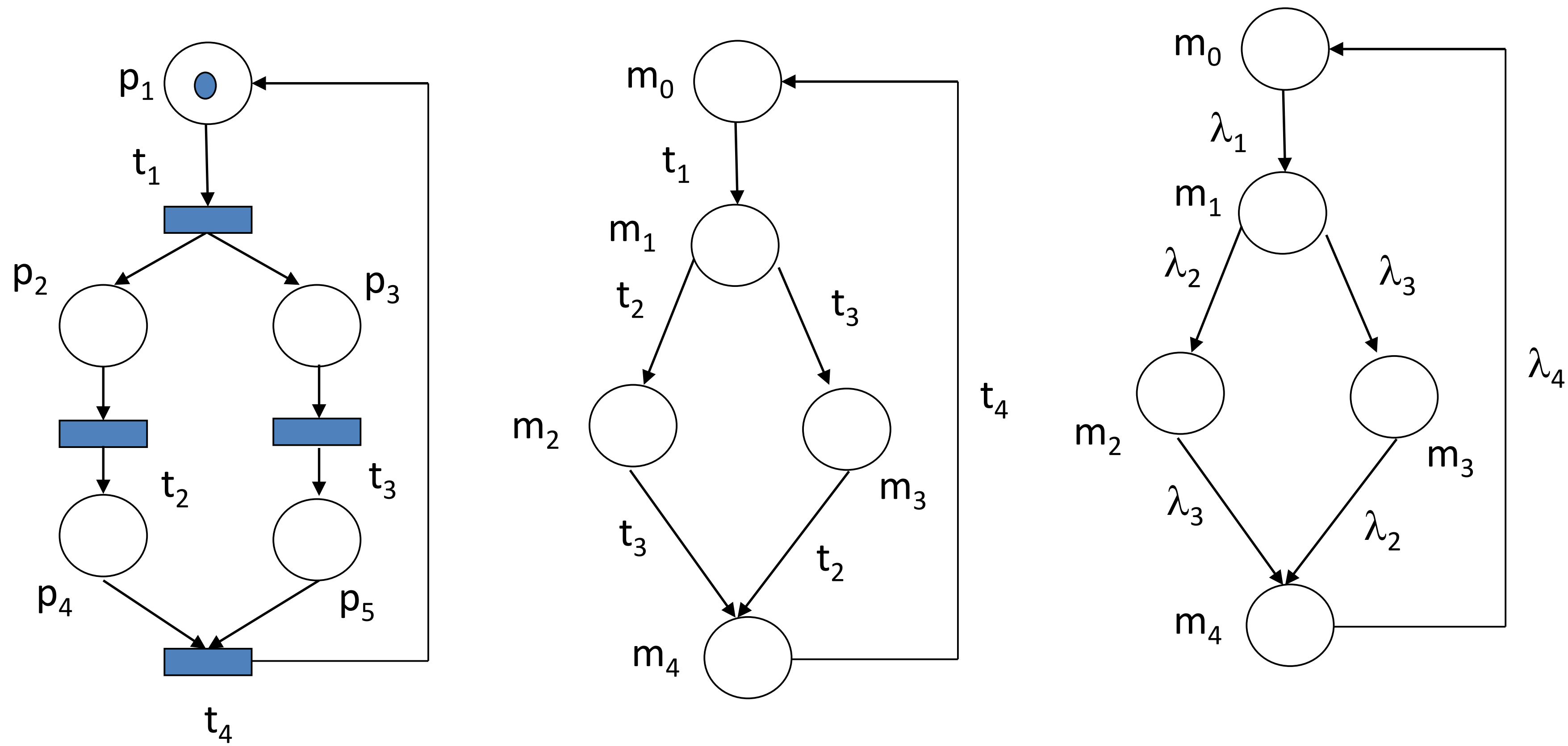
SPN example

Generating Reachability Graph:



$m_0=(1\ 0\ 0\ 0\ 0)^T$ $m_1=(0\ 1\ 1\ 0\ 0)^T$ $m_2=(0\ 0\ 1\ 1\ 0)^T$

$m_3=(0\ 1\ 0\ 0\ 1)^T$ $m_4=(0\ 0\ 0\ 1\ 1)^T$

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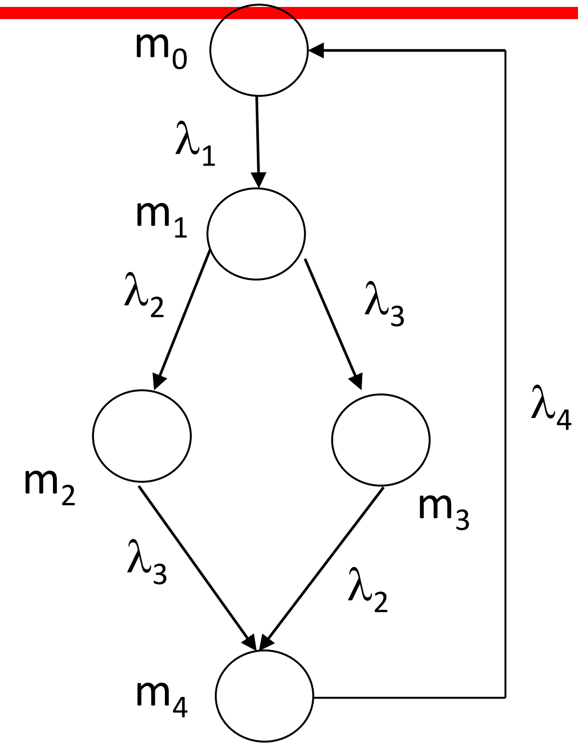
SPN example

Transition rate matrix A=?

$$\begin{bmatrix} a00 & a01 & a02 & a03 & a04 \\ a10 & a11 & a12 & a13 & a14 \\ a20 & a21 & a22 & a23 & a24 \\ a30 & a31 & a32 & a33 & a34 \\ a40 & a41 & a42 & a43 & a44 \end{bmatrix}$$



Decide **non-diagonal** elements first.

Then find diagonal elements according to

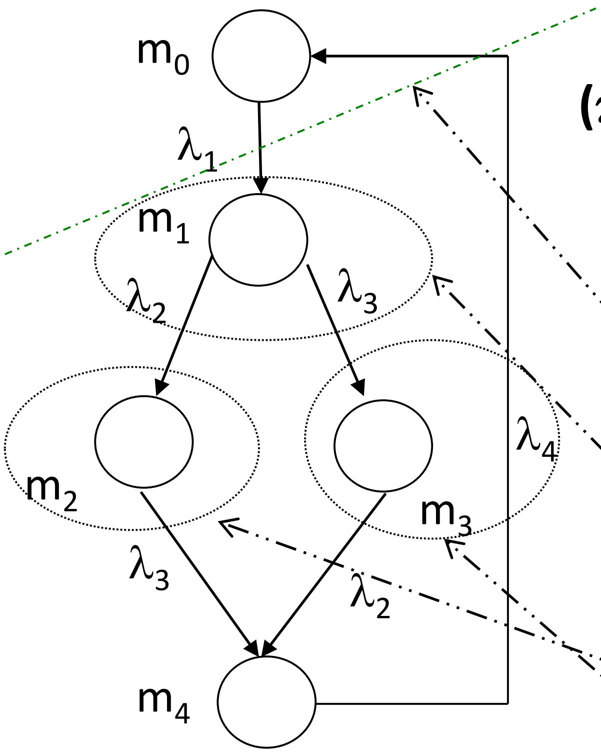
ai0+ai1+ai2+ai3+ai4=0, i=0, 1, 2, and 3.

For i≠j, aij=the total rate leaving from m_i to m_j.

$$\begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & 0 \\ 0 & -(\lambda_2+\lambda_3) & \lambda_2 & \lambda_3 & 0 \\ 0 & 0 & -\lambda_3 & 0 & \lambda_3 \\ 0 & 0 & 0 & -\lambda_2 & \lambda_2 \\ \lambda_4 & 0 & 0 & 0 & -\lambda_4 \end{bmatrix}$$

SPN example

Balance equations



$$(\pi 0, \pi 1, \pi 2, \pi 3, \pi 4) \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & 0 \\ 0 & -(\lambda_2+\lambda_3) & \lambda_2 & \lambda_3 & 0 \\ 0 & 0 & -\lambda_3 & 0 & \lambda_3 \\ 0 & 0 & 0 & -\lambda_2 & \lambda_2 \\ \lambda_4 & 0 & 0 & 0 & -\lambda_4 \end{pmatrix} = 0$$

$$-\lambda_1\pi_0 + \lambda_4\pi_4 = 0$$

$$\lambda_1\pi_0 - (\lambda_2+\lambda_3)\pi_1 = 0$$

$$\lambda_2\pi_1 - \lambda_3\pi_2 = 0$$

$$\lambda_3\pi_1 - \lambda_2\pi_3 = 0$$

SPN example

Produce Solution of Steady State Probabilities

$$-\lambda_1\pi_0 + \lambda_4\pi_4 = 0$$
$$\lambda_1\pi_0 - (\lambda_2+\lambda_3)\pi_1 = 0$$
$$\lambda_2\pi_1 - \lambda_3\pi_2 = 0$$
$$\lambda_3\pi_1 - \lambda_2\pi_3 = 0$$
$$\pi_0+\pi_1+\pi_2+\pi_3+\pi_4 = 1$$

1) Assume $\pi 0$ is known.
2) Represent others in terms of $\pi 0$ only.
3) Use the last equation to find $\pi 0$
4) Find $\pi 1$ to $\pi 4$.

given $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$,

$\pi 0 = \pi 4 = 2/7$

(The system is at either m0 or m4 with probability 2/7)

$\pi 1 = \pi 2 = \pi 3 = 1/7$

The system is at either m1, m2, or m3 with probability 1/7)

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Performance Measures

1) **Probability of a certain condition V being true.**

   Suppose G={m: The condition V is true at m in reachability graph}. Then

$$\text{Prob}(V) = \sum_{m_i \in G} \pi_i$$

Examples:

A machine utilization, i.e, a machine is in use;

A machine failure rate, i.e., a machine is in failure;

A channel idle rate, i.e., a channel is idle;

…

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Performance Measures

2) **Average number of tokens in a place p.**

Define condition V1: a token in p; V2: 2 tokens in p; …; Vk: k tokens in p. Assume p has at most k tokens.

E(m(p))=
Prob(V1)+2Prob(V2)+…+kProb(Vk)

$$= \sum_{i=1}^{k} i \times \mathrm{Prob}(Vi)$$

Performance Measures

3) **Frequency of firing a transition**, i.e., the average number of times the transition fires in unit time. Let G={m: tj is enabled at m in reachability graph}.

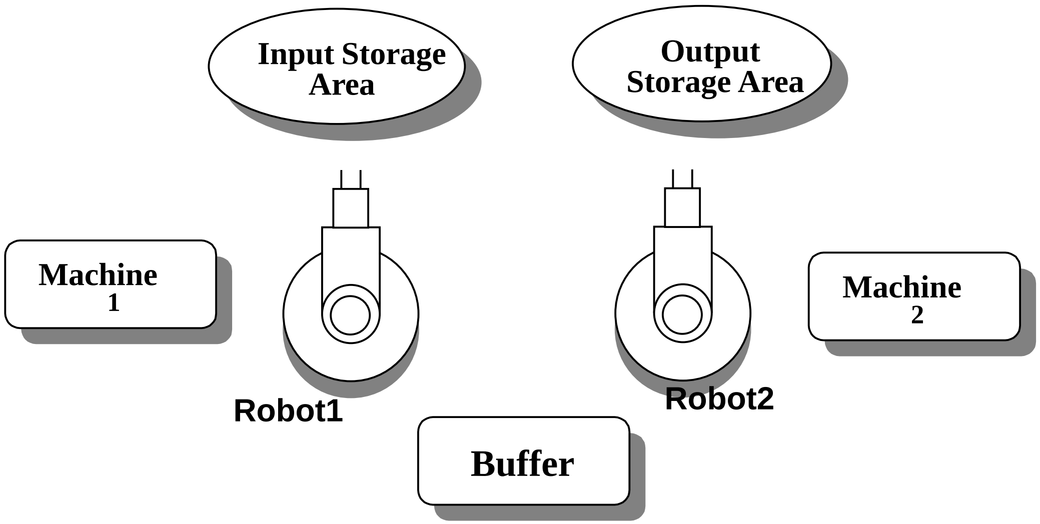
$$F(t_j) = \sum_{m_i \in G} \lambda_j(m_i) \times \pi_i$$

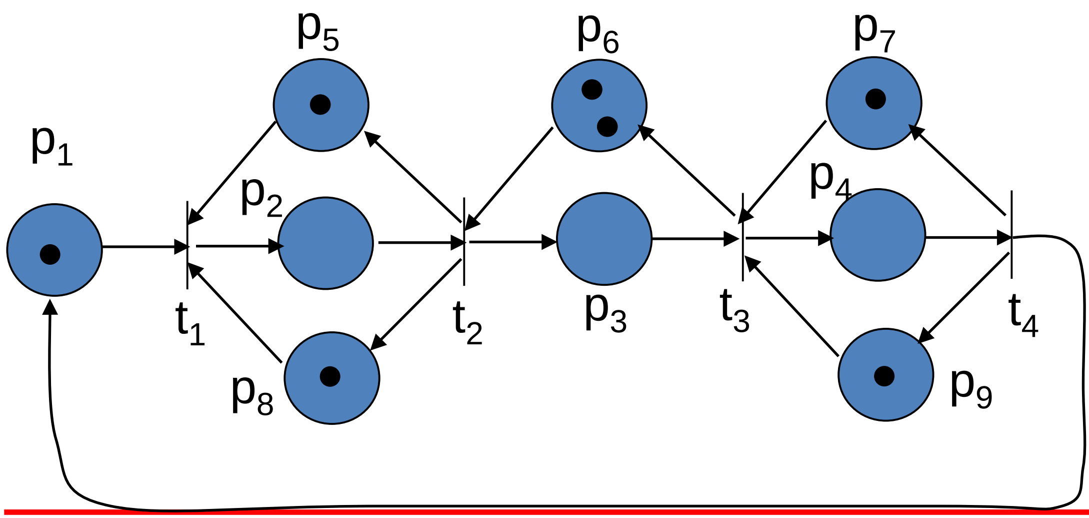
This measures the **system throughput**. Assume that firing each of t_a, t_b, .., t_c implies completion of a product, then the entire system's throughput is

$$F(t_a)+F(t_b)+\ldots+F(t_c).$$

Example: Manufacturing cell

An example we saw earlier in DTPN:





p1: Raw part available

p2: M1 processing a part (**10**)

p3: Intermediate parts ready

p4: M2 processing a part (**16**)

p5: M1 available

p6: buffer slots available

p7: M2 available

p8: R1 available

p9: R2 available

t1: R1 loading (**1**)

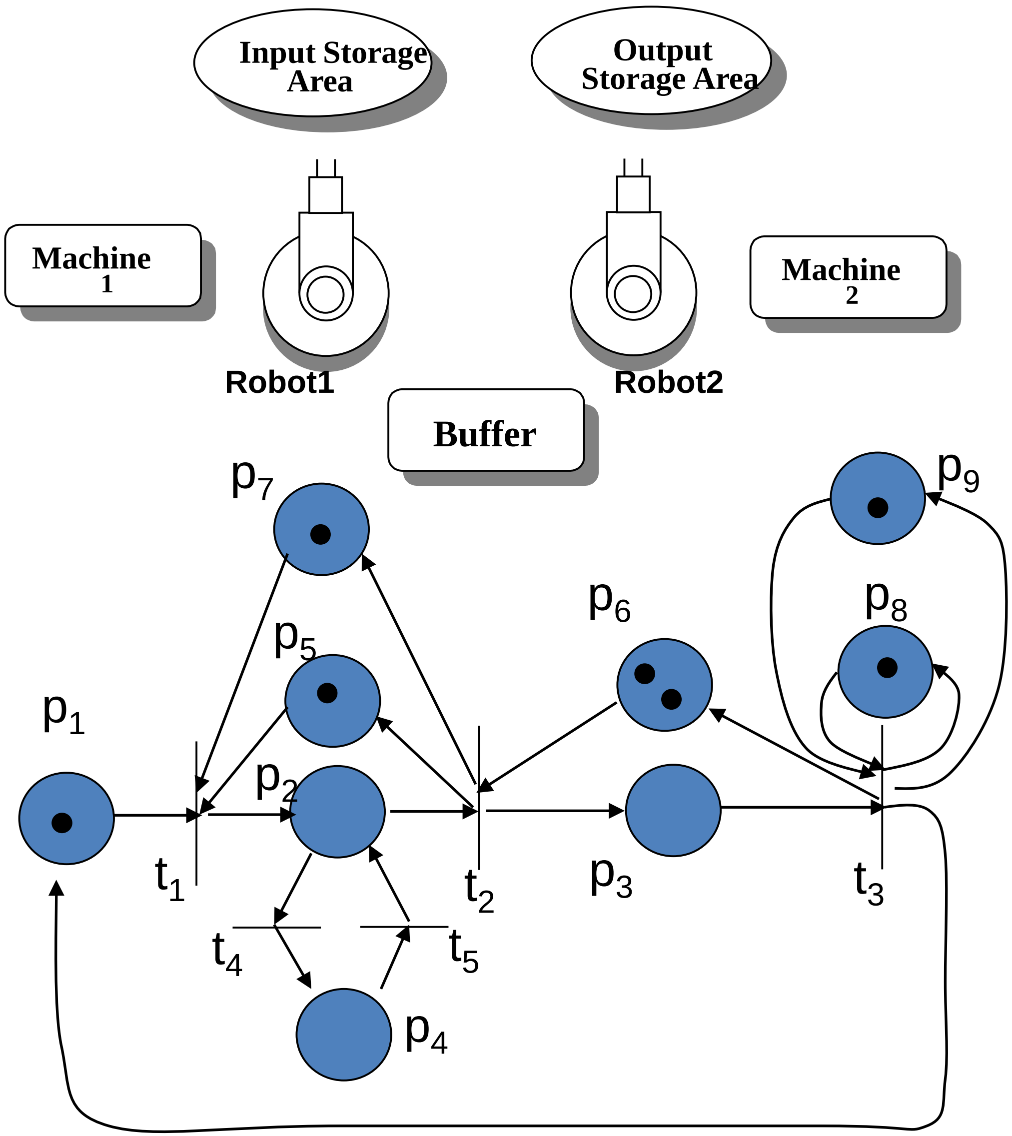
t2: R1 unloading (**1**)

t3: R2 loading (**1**)

t4: R2 unloading (**1**)

(t4,p1): load a raw part to input storage area (**2**)

Example: Manufacturing cell



- M1 may break down. On the average, M1 takes 2 time units to break down and ¼ time unit to be repaired. i.e. its average failure and repair rates (1/time unit) are 0.5 and 4.
- M2 is failure-free.
- R1's loading rate is 40. Average rate for M1's processing a part plus R1's unloading is 5 per unit time.
- Average rate for M2's processing plus the R2 loading and unloading is 4 per unit time.

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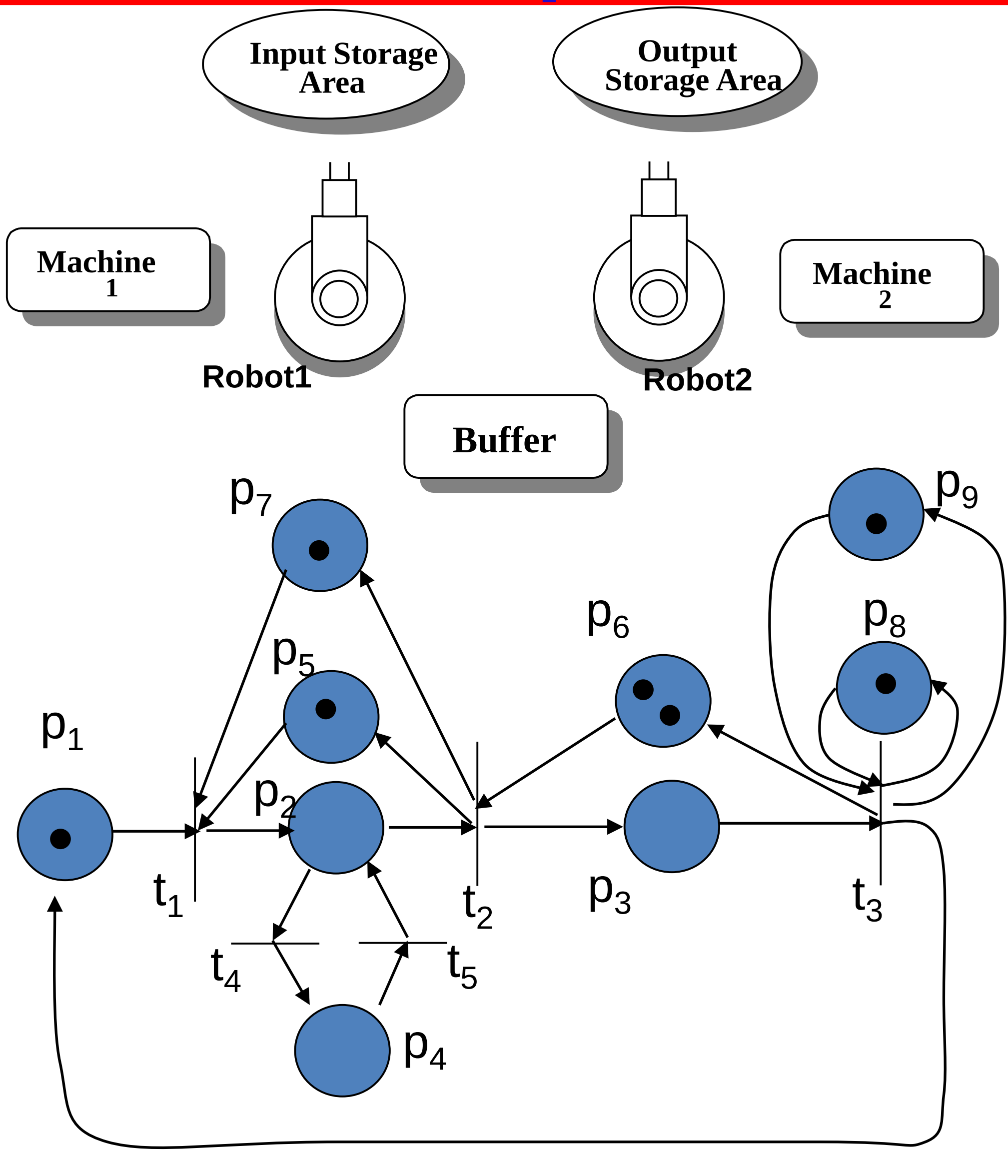
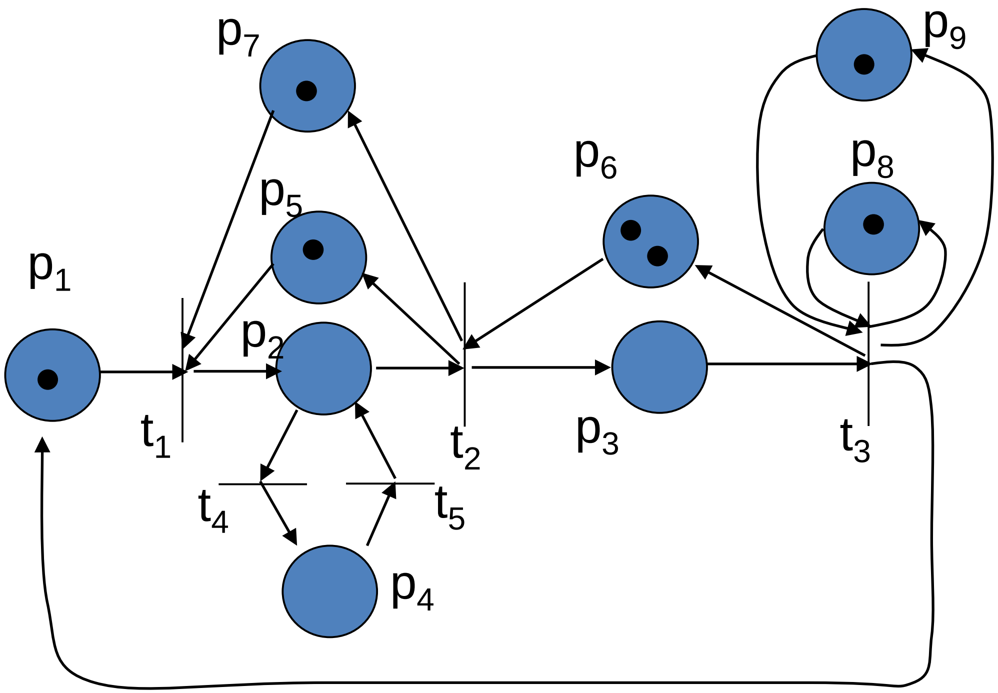
Example: Manufacturing cell

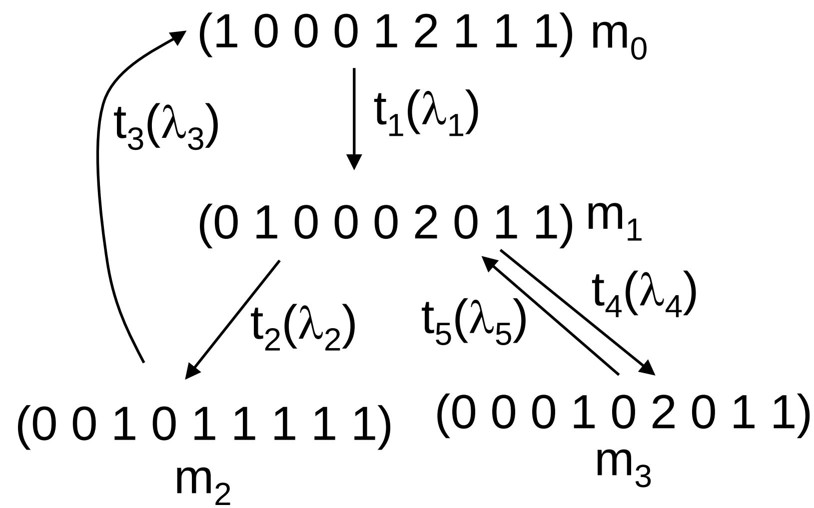


p1: parts are available
p2: M1 is in processing
p3: Part is ready for M2
p4: M1 is in repair
p5: M1 is available
p6: buffer is available
p7: R1 is available
p8: M2 is available
p9: R2 is available
t1: R1 is loading part to M1 (40)
t2: M1 is processing and R2 is unloading (5)
t3: M2 is processing, R2 is loading & unloading (4)
t4: M1 is breaking down (0.5)
t5: repair (4)

Example: Manufacturing cell

$\lambda_1=40;\ \lambda_2=5;\ \lambda_3=4;\ \lambda_4=0.5;\ \lambda_5=4$





# of states: 4

$$A=\begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ 0 & -\lambda_2-\lambda_4 & \lambda_2 & \lambda_4 \\ \lambda_3 & 0 & -\lambda_3 & 0 \\ 0 & \lambda_5 & 0 & -\lambda_5 \end{bmatrix}$$

What is the average utilization of M1 & system throughput?

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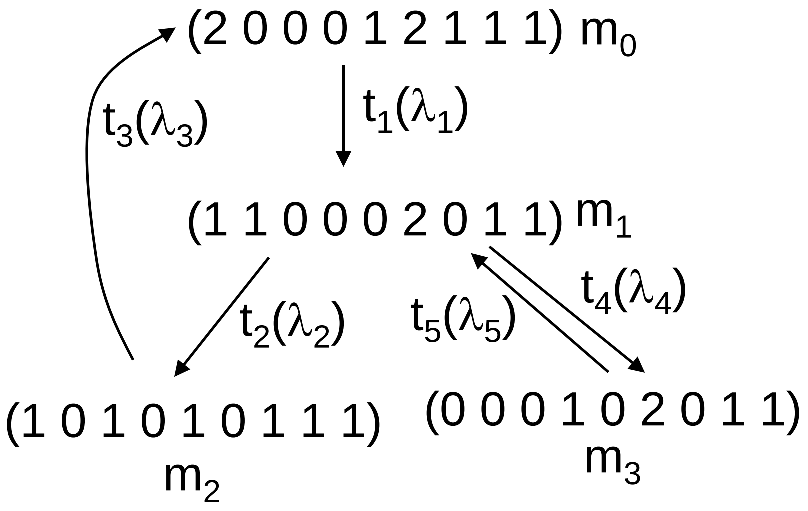

(2 0 0 0 1 2 1 1 1) m0
t1(λ1)
t3(λ3)
(1 1 0 0 0 2 0 1 1) m1
t2(λ2)
t5(λ5)
t4(λ4)
(1 0 1 0 1 0 1 1 1)
m2
(0 0 0 1 0 2 0 1 1)
m3

Example: Manufacturing cell

Steady-state probabilities can be obtained by solving the following equations:

$$[\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]A = 0$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$where\ A = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ 0 & -\lambda_2 - \lambda_4 & \lambda_2 & \lambda_4 \\ \lambda_3 & 0 & -\lambda_3 & 0 \\ 0 & \lambda_5 & 0 & -\lambda_5 \end{bmatrix}$$

We obtain:

$$\pi_0 = \lambda_2\lambda_3\lambda_5 / \lambda$$

$$\pi_1 = \lambda_1\lambda_3\lambda_5 / \lambda$$

$$\pi_2 = \lambda_1\lambda_2\lambda_5 / \lambda$$

$$\pi_3 = \lambda_1\lambda_3\lambda_4 / \lambda$$

$$\lambda = \lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4$$

λ_1=40; λ_2=5; λ_3=4; λ_4=0.5; λ_5=4

$$\pi_0 = 0.05$$

$$\pi_1 = 0.4$$

$$\pi_2 = 0.5$$

$$\pi_3 = 0.05$$

What is the average utilization of M1 & system throughput?

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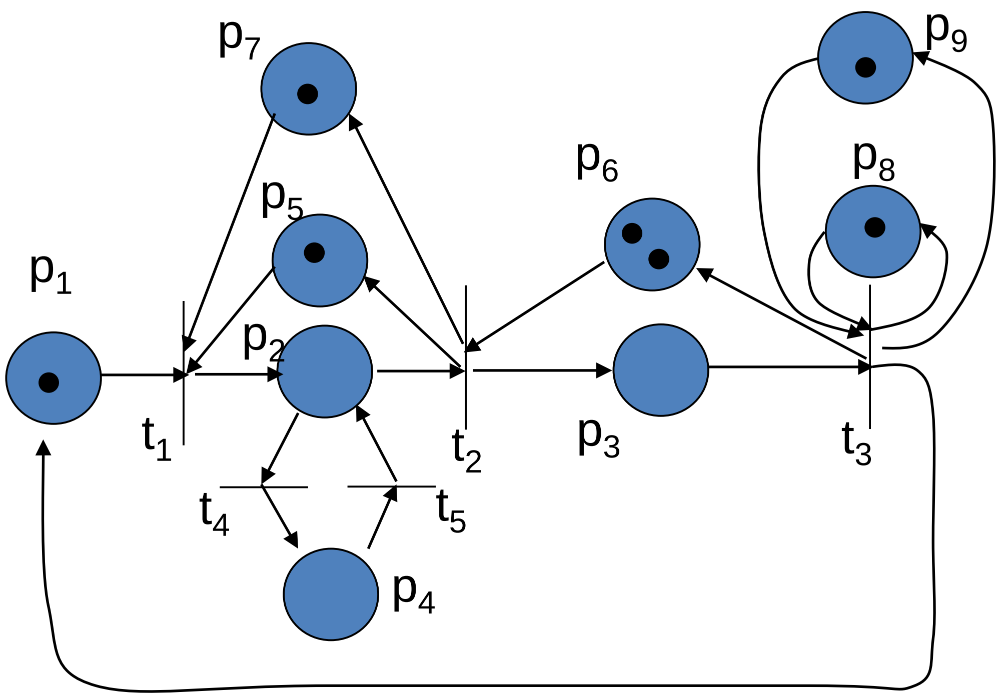
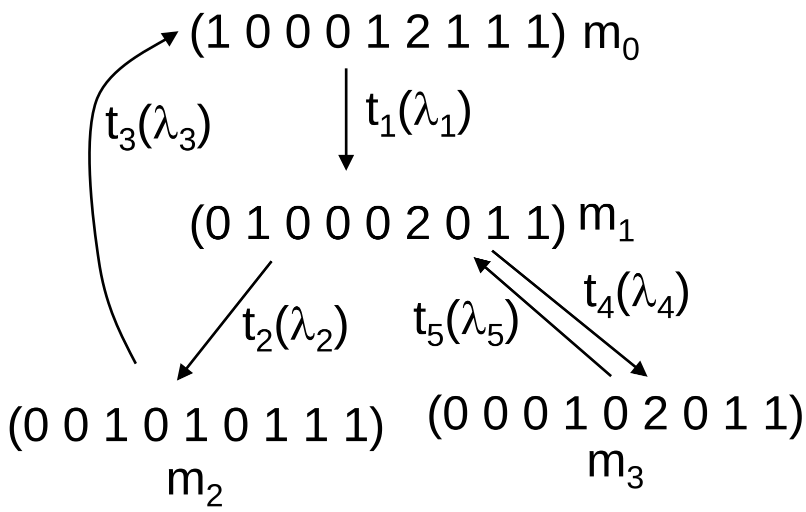
Example: Manufacturing cell

What is the average utilization of M1?



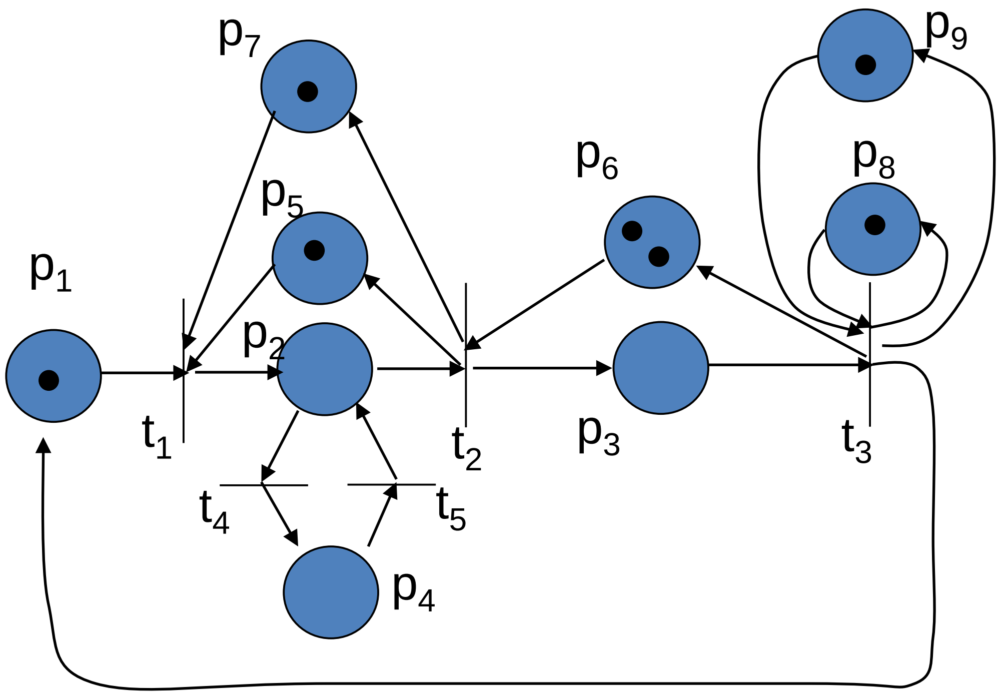


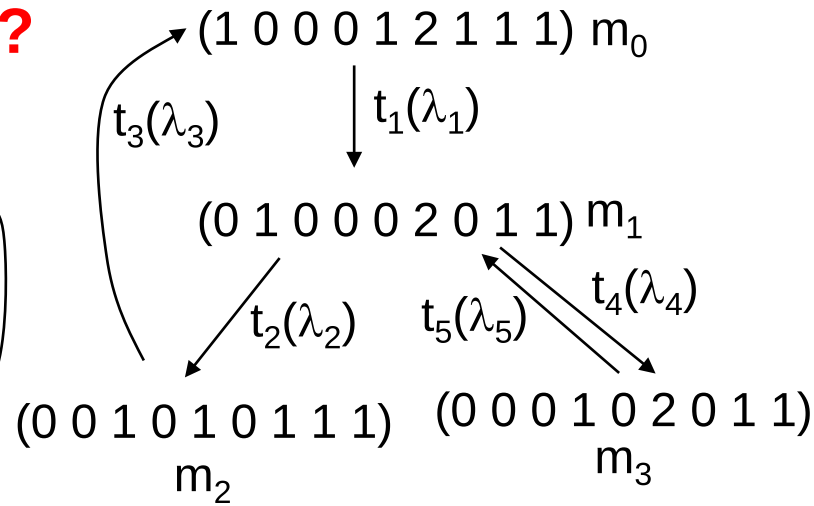
$$\pi_0 = 0.05$$
$$\pi_1 = 0.4$$
$$\pi_2 = 0.5$$
$$\pi_3 = 0.05$$

The average utilization of M1 ⇔ probability that M1 is in process ⇔ probability of P2 being marked ⇔ probability of being at marking m1 = π_1=0.4=40%

Example: Manufacturing cell

What is the system throughput?



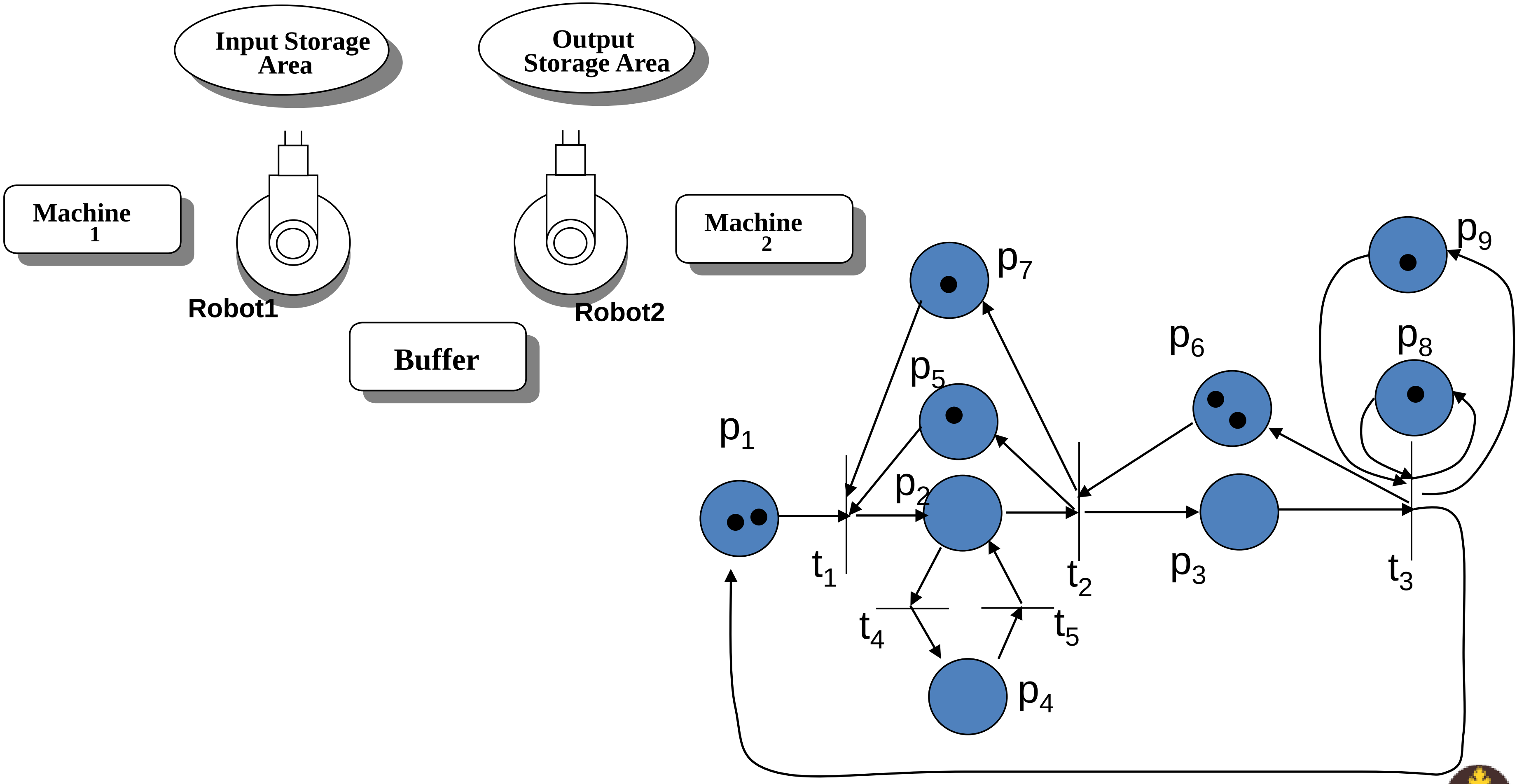


$$\pi_0 = 0.05$$
$$\pi_1 = 0.4$$
$$\pi_2 = 0.5$$
$$\pi_3 = 0.05$$

The completion of a product is modeled by firing t3. t3 is enabled at m2. System throughput = $\pi_2 * \lambda_3 = 0.5*4 = 2$ workpieces/unit time

Exercise: Manufacturing cell

What is the average utilization of M1 and the system throughput if two parts are in the input area?



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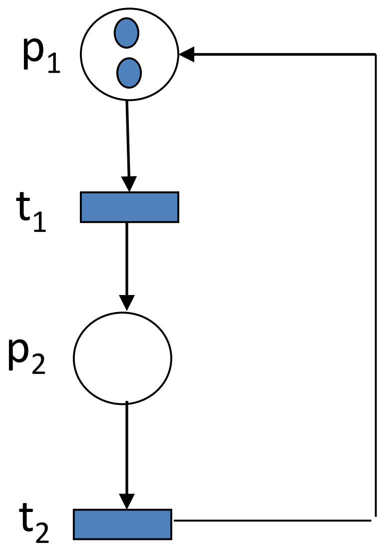
Marking-dependent transition firing rate



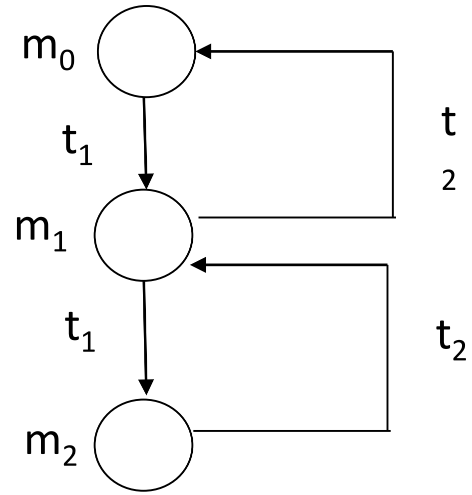
$m_0=(2\ 0)^T$

$t_1 \downarrow \uparrow t_2$

$m_1=(1\ 1)^T$

$t_1 \downarrow \uparrow t_2$

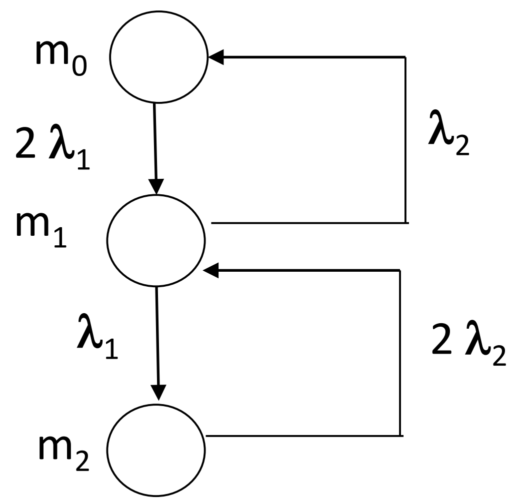
$m_2=(0\ 2)^T$



Assume that

t1's firing rate is $m(p_1)\lambda_1$

t2's firing rate is $m(p_2)\lambda_2$



Marking-dependent transition firing rate

Throughput Performance of the example:

- Solving the balance equations leads to steady state probabilities, π_0, π_1 and π_2.
- Then if firing t_2 means producing an item, the throughput is: $\lambda_2\pi_1 + 2\lambda_2\pi_2$
- The average # of tokens in p_1 is $2\pi_0+\pi_1$
- The average # of tokens in p_2 is $\pi_1+2\pi_2$