ECE 09468/09568

Discrete Event Systems

Lecture 7: **Petri Nets (PNs) – Part III**

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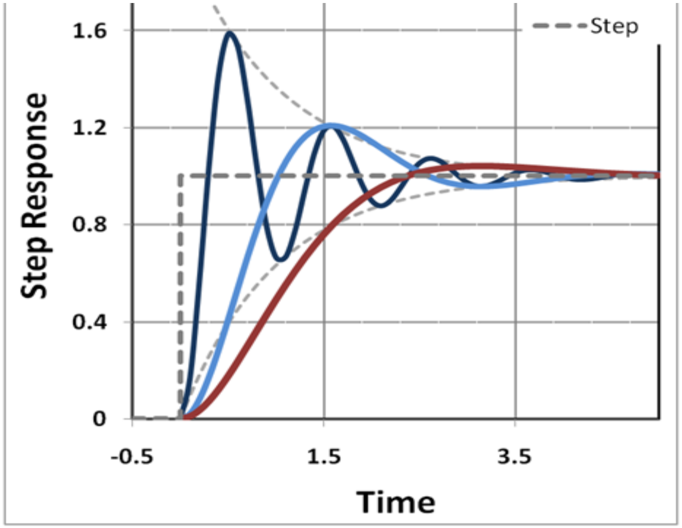
Analysis Approaches of PN

- The dynamic behavior of many systems studied in engineering can be described by differential equations or algebraic equations.

e.g. A continuous system:

$$x'(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$



- Can some equations be used to study behavior and properties of a DES?

Incidence Matrix Approach

- Uses **incidence matrix** to study the structural behavior of PN
- It defines all possible interconnections between places and transitions.
- Incidence Matrix "A" of a PN is of dimension:

  (# of places) X (# of transitions) or |P| X |T|

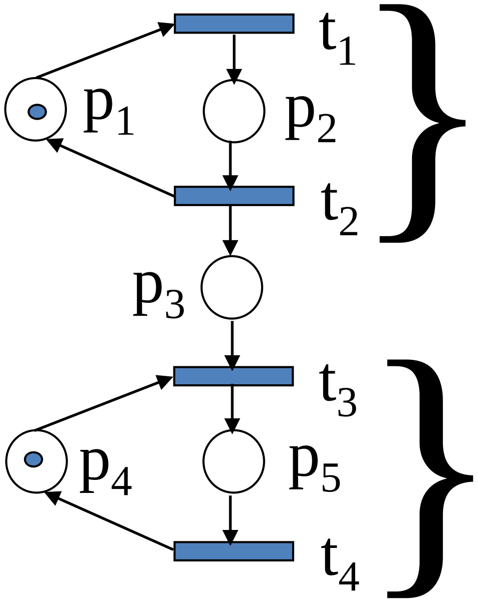
- An entry of A is defined as $a_{ij} = a_{ij}^{+} - a_{ij}^{-}$ where:
  - a_{ij}^{+} is equal to the number of arcs connecting transition *j* to its output place *i* and
  - a_{ij}^{-} is equal to the number of arcs connecting transition *j* from its input place *i.*

Incidence Matrix Approach

- A=O-I where O and I are output and input functions of PN.
- When transition *j* fires,
  - a_{ij}^{+} represents the number of token deposited on its output place p_i
  - a_{ij}^{-} represents the number of tokens removed from its input place p_i
  - a_{ij} represents change in the number of tokens in place i when transition t_j fires once

Incidence Matrix Approach

Example: **a PN model for Producing & Consuming process**



Production Part

Consumption Part

$$I = \begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & 1 & 0 & 0 & 0 \\ p2 & 0 & 1 & 0 & 0 \\ p3 & 0 & 0 & 1 & 0 \\ p4 & 0 & 0 & 1 & 0 \\ p5 & 0 & 0 & 0 & 1 \end{array}$$

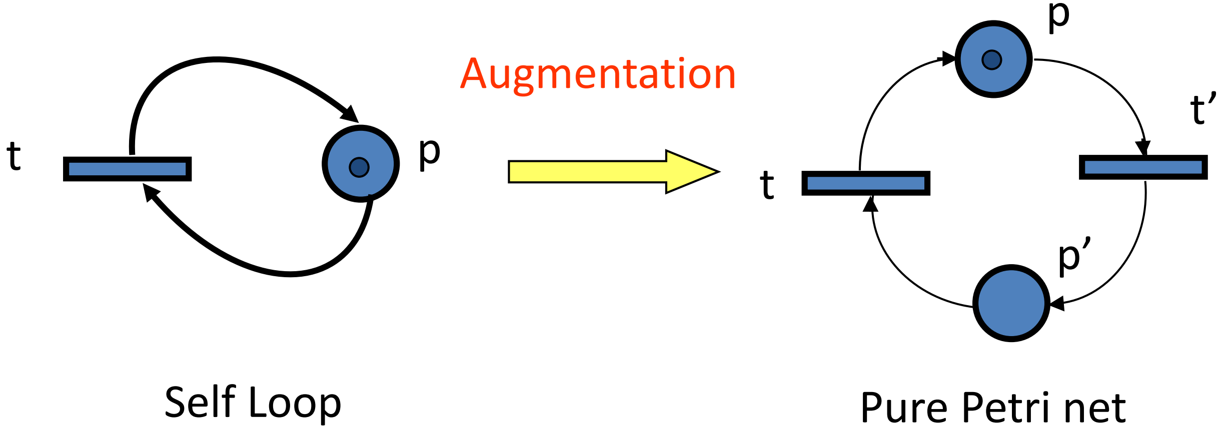
$$O = \begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & 0 & 1 & 0 & 0 \\ p2 & 1 & 0 & 0 & 0 \\ p3 & 0 & 1 & 0 & 0 \\ p4 & 0 & 0 & 0 & 1 \\ p5 & 0 & 0 & 1 & 0 \end{array}$$

$$A = O - I = \begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & -1 & 1 & 0 & 0 \\ p2 & 1 & -1 & 0 & 0 \\ p3 & 0 & 1 & -1 & 0 \\ p4 & 0 & 0 & -1 & 1 \\ p5 & 0 & 0 & 1 & -1 \end{array}$$

Incidence Matrix Approach

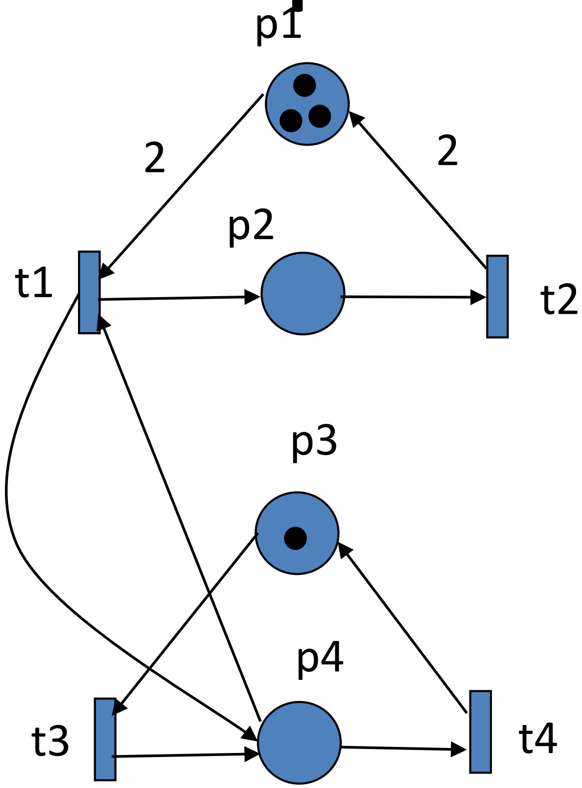
Self-loop:

- Incidence Matrix can reflect the structure of a **pure** PN only
- For a PN with self-loops, $a_{ij} = 0$ for place *i* and transition *j* that belong to a self loop.
- For incidence matrix to reflect the structure of PN, all self-loops, if any, should be replaced by two additional places and a transition



Incidence Matrix Approach

More Examples:



$$I = \begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & 2 & 0 & 0 & 0 \\ p2 & 0 & 1 & 0 & 0 \\ p3 & 0 & 0 & 1 & 0 \\ p4 & 1 & 0 & 0 & 1 \end{array}$$

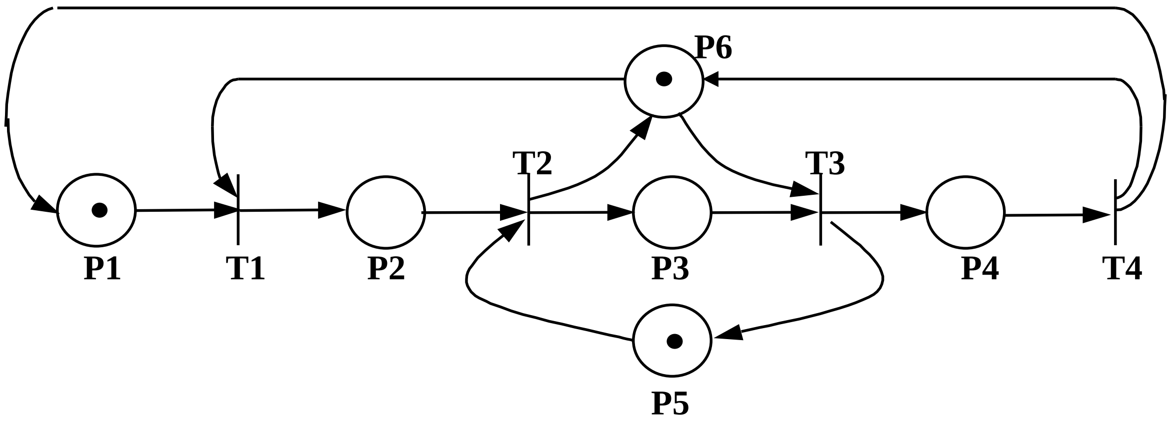
$$O = \begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & 0 & 2 & 0 & 0 \\ p2 & 1 & 0 & 0 & 0 \\ p3 & 0 & 0 & 0 & 1 \\ p4 & 1 & 0 & 1 & 0 \end{array}$$

$$A = O - I$$

$$A = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Incidence Matrix Approach

More Examples:



$$I=\begin{array}{c|cccc|} & T1 & T2 & T3 & T4 \\ P1 & 1 & 0 & 0 & 0 \\ P2 & 0 & 1 & 0 & 0 \\ P3 & 0 & 0 & 1 & 0 \\ P4 & 0 & 0 & 0 & 1 \\ P5 & 0 & 1 & 0 & 0 \\ P6 & 1 & 0 & 1 & 0 \end{array}$$

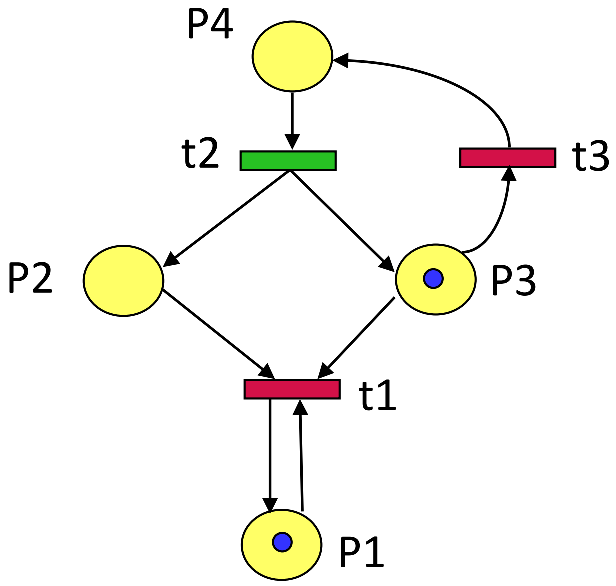
$$O=\begin{array}{c|cccc|} & T1 & T2 & T3 & T4 \\ P1 & 0 & 0 & 0 & 1 \\ P2 & 1 & 0 & 0 & 0 \\ P3 & 0 & 1 & 0 & 0 \\ P4 & 0 & 0 & 1 & 0 \\ P5 & 0 & 0 & 1 & 0 \\ P6 & 0 & 1 & 0 & 1 \end{array}$$

$$A=O\text{-}I=\begin{array}{c|cccc|} & T1 & T2 & T3 & T4 \\ P1 & -1 & 0 & 0 & 1 \\ P2 & 1 & -1 & 0 & 0 \\ P3 & 0 & 1 & -1 & 0 \\ P4 & 0 & 0 & 1 & -1 \\ P5 & 0 & -1 & 1 & 0 \\ P6 & -1 & 1 & -1 & 1 \end{array}$$

Incidence Matrix Approach

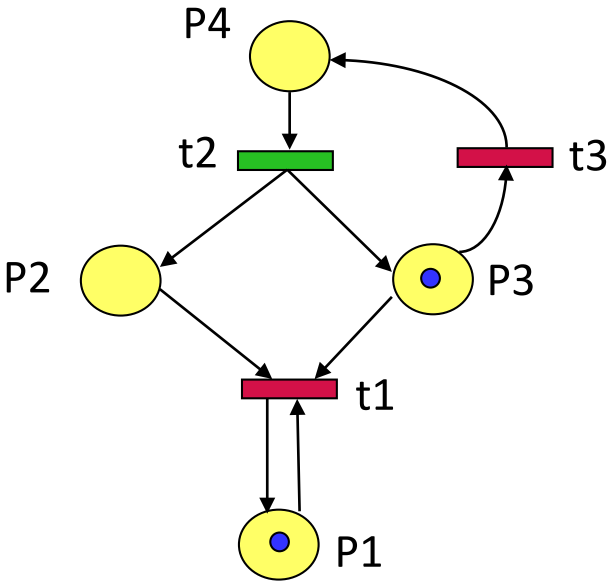
Your turns…

Find Incidence Matrix for the following net:



Incidence Matrix Approach

Find Incidence Matrix for the following net



$$I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A = O - I =} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Incidence Matrix Approach

State Equation

- A change in marking due to transition firing can be represented in an equation.

  $M' = M + A\mu$ where

  - A is the incidence matrix
  - μ is an nx1 column vector having only one at its i-th position representing transition i firing, n is # of transitions.
  - M’ & M are mx1 column vectors representing a marking. M is current marking and M’ is next marking, m is # of places.

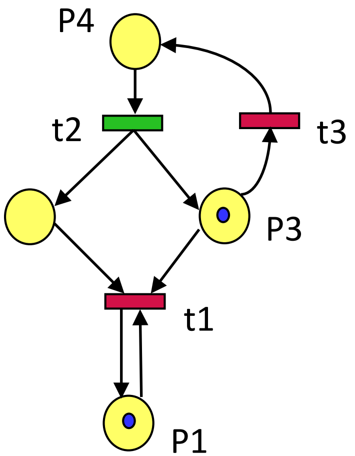
Incidence Matrix Approach

State Equation (cont.)

- A change in marking due to transition firing can be represented in an equation.

$$M' = M + A\mu$$

e.g. Find new marking after firing transition t_3



$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mu = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$M' = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} X \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Incidence Matrix Approach

Vectors and Their Operations

T is called "Transpose"

Let $x=(x1, x2, x3)^T$ and $y=(y1, y2, y3)^T$.

Vector addition: $x+y= (x1+y1, x2+y2, x3+y3)^T$

Vector subtraction: $x-y= (x1-y1, x2-y2, x3-y3)^T$

Inner product of two vectors: $x^Ty= x1\times y1+x2\times y2+x3\times y3$

xy^T is a matrix:

$$\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} (y1, y2, y3) = \begin{pmatrix} x1y1 & x1y2 & x1y3 \\ x2y1 & x2y2 & x2y3 \\ x3y1 & x3y2 & x3y3 \end{pmatrix}$$

Row vector

Column vector

Incidence Matrix Approach

Vectors and Their Operations (cont.)

Let $x=(x1, x2, x3)^T$ and a matrix

$$A = \begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}$$

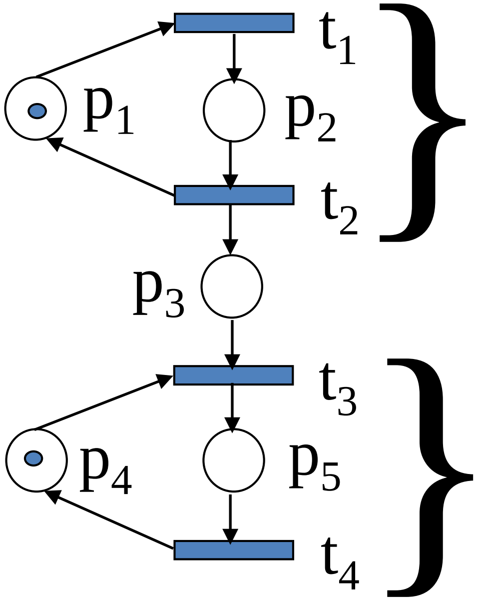
What is Ax=?

$$\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \end{pmatrix}\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} a11x1+a12x2+a13x3 \\ a21x1+a22x2+a23x3 \end{pmatrix}$$

NOTE: For the above given A and x,
xA, Ax^T, and x^TA are all undefined.

Incidence Matrix Approach

Example: Find out the marking after t1 fires from marking(1,0,0,1,0)



Production Part

Consumption Part

$$A=O-I=\begin{array}{c|cccc} & t1 & t2 & t3 & t4 \\ \hline p1 & -1 & 1 & 0 & 0 \\ p2 & 1 & -1 & 0 & 0 \\ p3 & 0 & 1 & -1 & 0 \\ p4 & 0 & 0 & -1 & 1 \\ p5 & 0 & 0 & 1 & -1 \end{array}$$

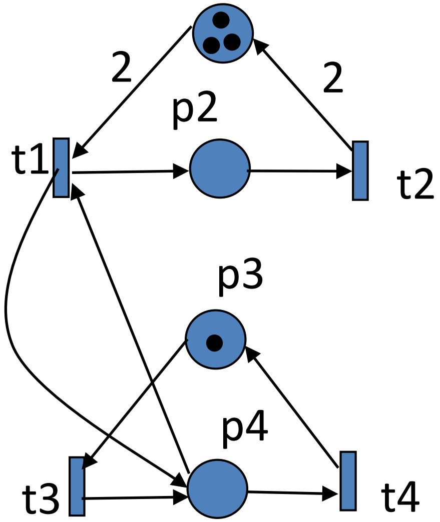
$$\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$M' = M + A\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{X} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Incidence Matrix Approach

Your turns…

Find the marking after firing t3 at M_0 (3 0 1 0):

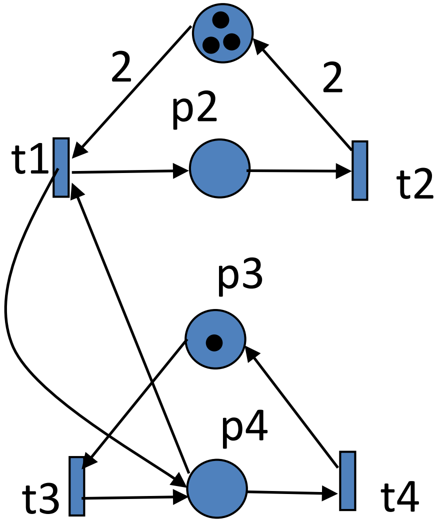


$$A = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Incidence Matrix Approach

Find the marking after firing t3 at M $(3\ 0\ 1\ 0)^T$:

$$A = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

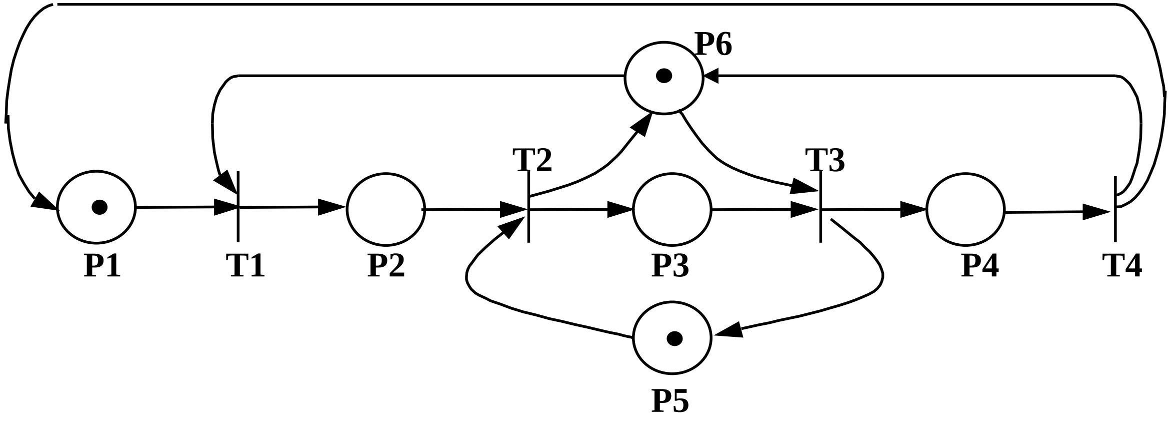


$$M' = M + A\mu = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} X \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Incidence Matrix Approach

More Examples:

Find marking after firing T1:

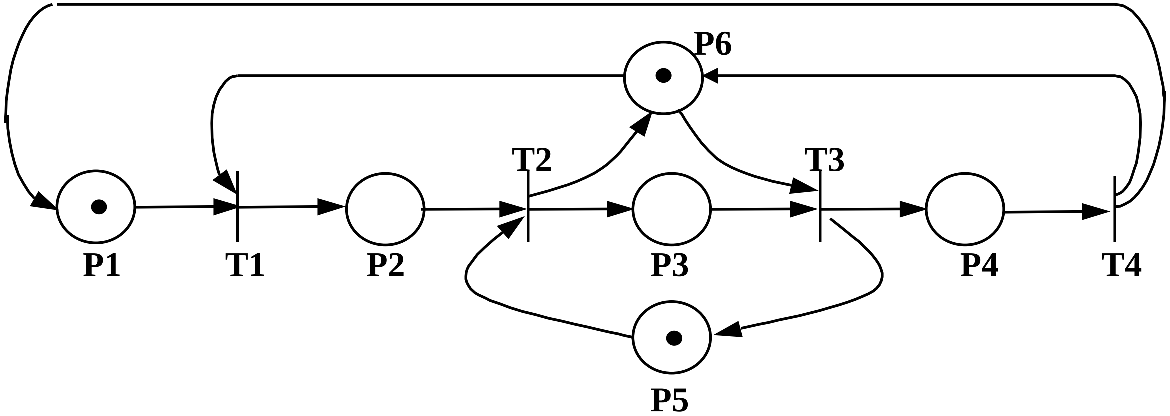


$$A=O\text{-}I= \begin{array}{c|cccc} & T1 & T2 & T3 & T4 \\ \hline P1 & -1 & 0 & 0 & 1 \\ P2 & 1 & -1 & 0 & 0 \\ P3 & 0 & 1 & -1 & 0 \\ P4 & 0 & 0 & 1 & -1 \\ P5 & 0 & -1 & 1 & 0 \\ P6 & -1 & 1 & -1 & 1 \end{array}$$

Incidence Matrix Approach

Find marking after firing T1:

$$A=\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$



$$M' = M + A\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} X \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Incidence Matrix Approach

State Equation

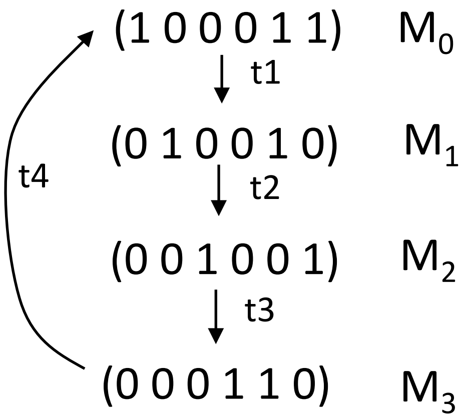
- The state equation for a Petri net represents a change in the distribution of tokens on places (marking) due to transition firing.

$$M_k = M_{k-1} + A\mu_k \text{ , } k = 1, 2, \ldots \text{ where}$$

  - M_k is an mx1 column vector representing a marking M_k immediately reachable from M_{k-1} after firing transition i.
  - μ_k is an nx1 column vector having only one at its i-th position representing transition i firing in the k-th firing.

Incidence Matrix Approach

Example: **Find marking after firing T1:**



(1 0 0 0 1 1) M_0

t1

(0 1 0 0 1 0) M_1

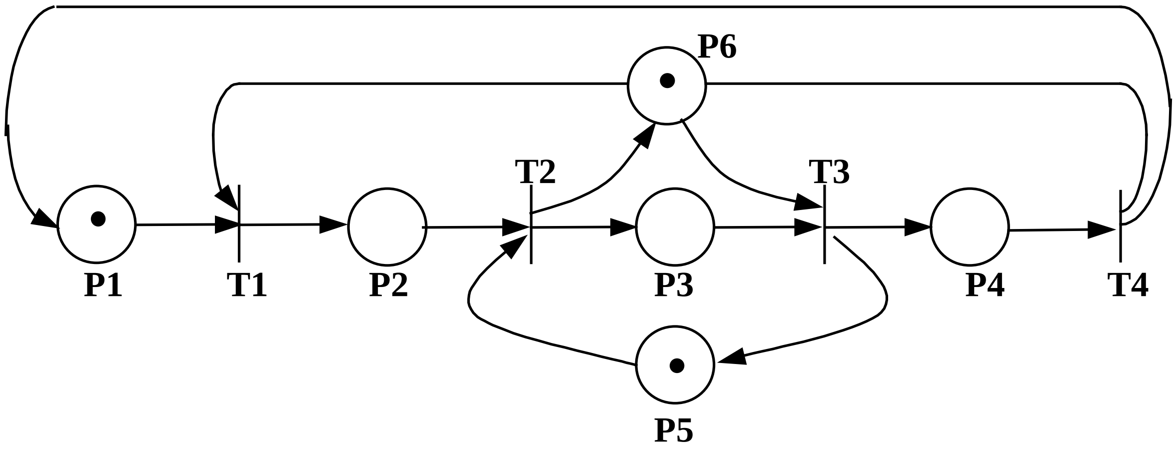
t2

(0 0 1 0 0 1) M_2

t3

(0 0 0 1 1 0) M_3

t4



$$M_1 = M_0 + A\mu_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \text{X} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mu_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mu_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Incidence Matrix Approach

State Equation

- State equation can be extended to firing a sequence of transitions.
- The change in marking due to transitions' firing can be represented in an equation.

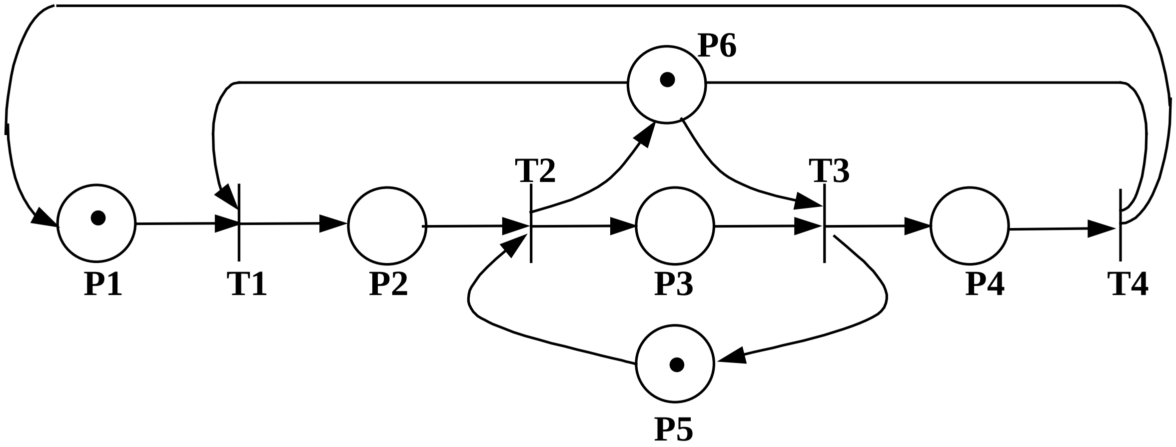
$$M' = M + A\mu \quad \text{where}$$

  - μ is an nx1 column vector. A positive number at its i-th position representing transition i firing. The positive number represents # of times the transition fire. μ is called firing vector.
  - M' is a marking reachable from M by firing a sequence of transitions.

Incidence Matrix Approach

Example: **Find marking after firing sequence of T1, T2, T3, T4:**

$$\begin{tikzcd}
(1\ 0\ 0\ 0\ 1\ 1) \arrow[d, "t1"] & M_0 \\
(0\ 1\ 0\ 0\ 1\ 0) \arrow[d, "t2"] & M_1 \\
(0\ 0\ 1\ 0\ 0\ 1) \arrow[d, "t3"] & M_2 \\
(0\ 0\ 0\ 1\ 1\ 0) \arrow[uuu, bend left=60, "t4"] & M_3
\end{tikzcd}$$



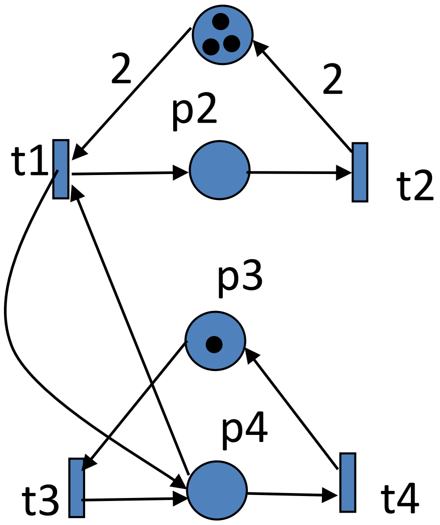
$$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M' = M_0 + A\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \text{X} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Incidence Matrix Approach

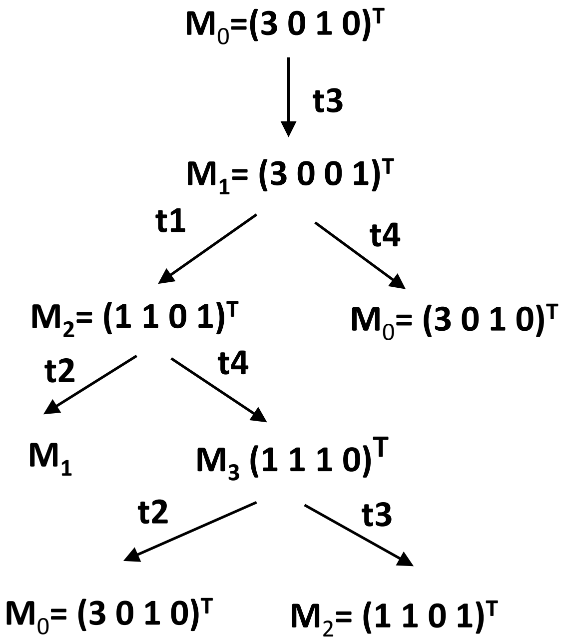
More Example

Find the marking after firing t3 and t4:



$$A = \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$



$M_0=(3\ 0\ 1\ 0)^T$

t3

$M_1= (3\ 0\ 0\ 1)^T$

t1

t4

$M_2= (1\ 1\ 0\ 1)^T$

$M_0= (3\ 0\ 1\ 0)^T$

t2

t4

M_1

$M_3\ (1\ 1\ 1\ 0)^T$

t2

t3

$M_0= (3\ 0\ 1\ 0)^T$

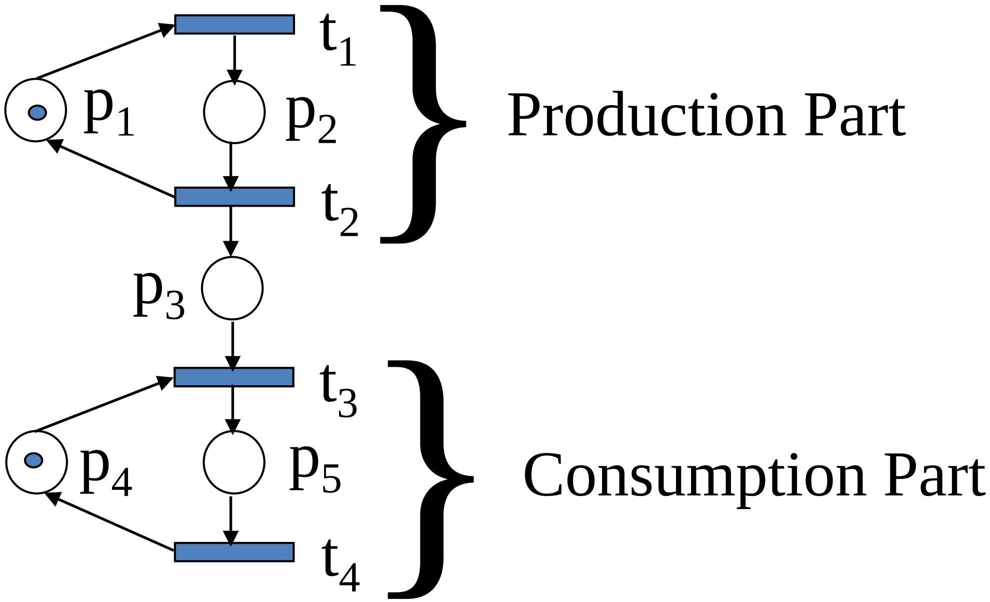
$M_2= (1\ 1\ 0\ 1)^T$

$$\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} X \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Incidence Matrix Approach

Your turns…

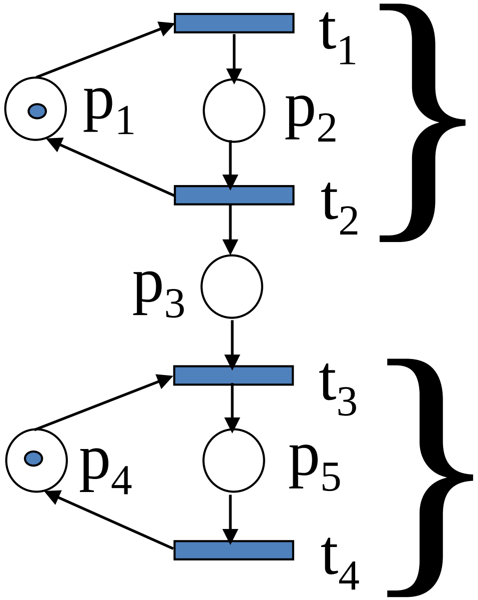
e.g. Using Matrix Equation approach, find out the marking after firing t1, t2, t3, t4 each once.



$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Exercise Problem Solution

e.g. find out the marking after firing t1, t2, t3, t4 each once.



Production Part

Consumption Part

$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

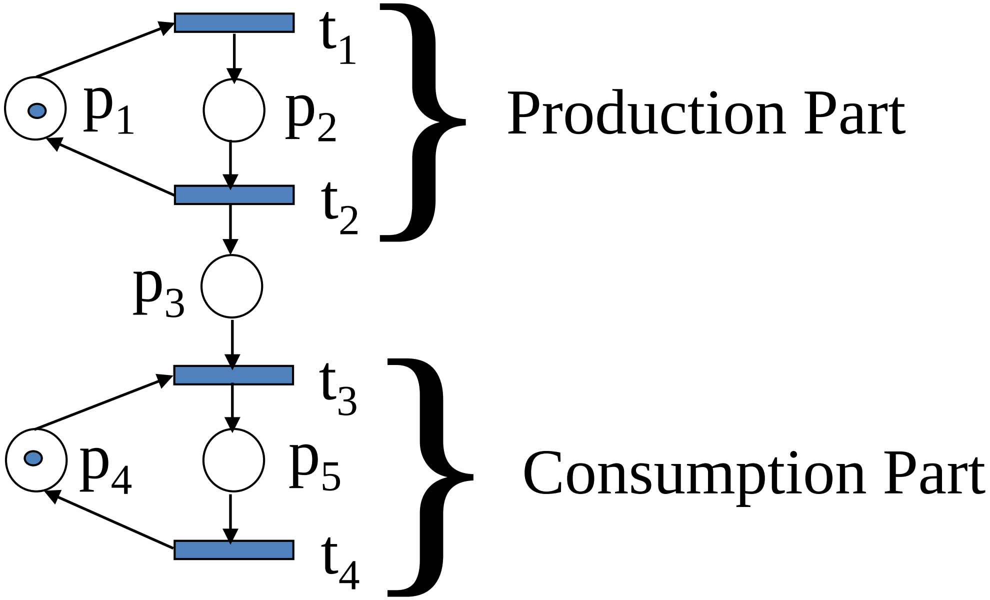
$$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M' = M + A\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \mathrm{X} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Incidence Matrix Approach

Your turns…

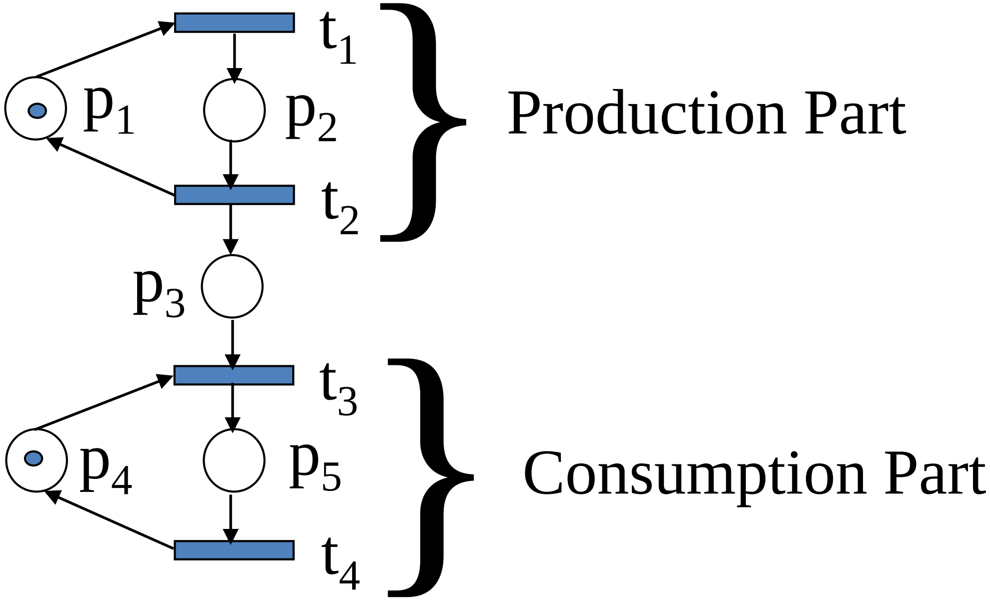
Find out the marking after firing t1, t2 three times and t3, t4 once.



$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Exercise Problem Solution

e.g. find out the marking after firing t1, t2 three times and t3, t4 once.



$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\mu = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

$$M' = M + A\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} X \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Invariants

- Incidence matrix is useful to find **structural properties** (independent of the initial marking).
- It can be used to find P-invariants and T-invariants
- A non-negative integer solution μ of Aμ = 0 is **T-invariant.**
- **T-invariants: Set of transitions that when fired returns the net to its previous marking.** Related to **reversibility.**
- A non-negative integer solution x of $A^T x=0$ is **P-invariant.**
- **P-invariants keep their weighted marking unchanged**. Related to **boundedness** and **conservativeness**.

T-invariant

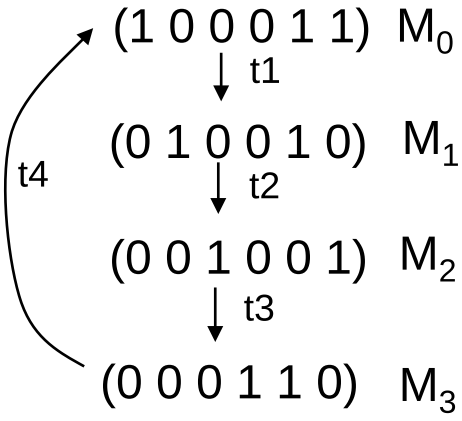
- The change in marking due to transitions' firing can be represented in an equation.

$$M' = M + A\mu$$

- $M = M + A\mu$ means that the system goes back to previous marking M after the firing of a sequence of transitions that defined by μ.
- A non-negative integer solution μ of $A\mu = 0$ is **T-invariant.**
- T-invariants show the cyclic behavior of a PN.

Example: T-invariant

Find T-invariant:



$(1\ 0\ 0\ 0\ 1\ 1)\ M_0$

$\downarrow t1$

$(0\ 1\ 0\ 0\ 1\ 0)\ M_1$

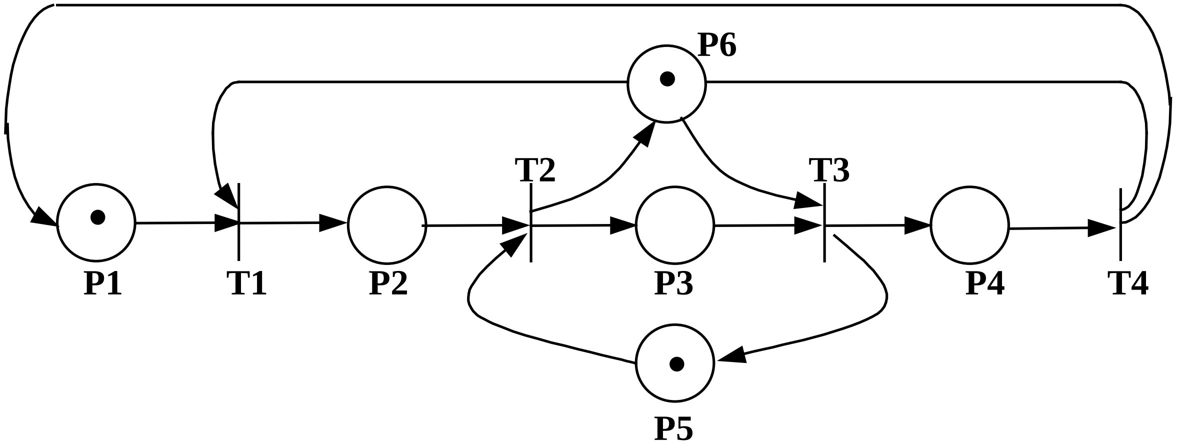
$\downarrow t2$

$(0\ 0\ 1\ 0\ 0\ 1)\ M_2$

$\downarrow t3$

$(0\ 0\ 0\ 1\ 1\ 0)\ M_3$

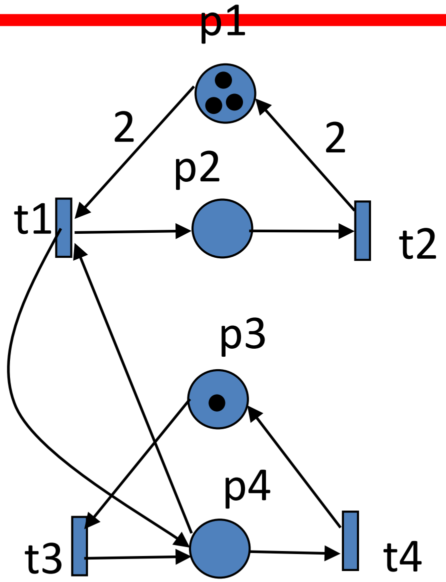
t4



$$A\mu = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \times \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} -\mu_1 + \mu_4 = 0 \\ \mu_1 - \mu_2 = 0 \\ \mu_2 - \mu_3 = 0 \\ \mu_3 - \mu_4 = 0 \\ -\mu_2 + \mu_3 = 0 \\ -\mu_1 + \mu_2 - \mu_3 + \mu_4 = 0 \end{cases}$$

$\mu_1 = \mu_2 = \mu_3 = \mu_4$ $\qquad$ $\mu = [1\ 1\ 1\ 1]^T$

Example: T-invariant



$$A\,y = 0 \quad \longrightarrow \quad \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0$$

Solving the above equations gives

$-2y_1 + 2y_2 = 0 \rightarrow y_2 = y_1$

$-y_3 + y_4 = 0 \rightarrow y_4 = y_3$

Let free variables $y_1 = y_3 = 1$

We have a <u>positive</u> T-invariant is **y=**(1 1 1 1)$^{\tau}$.

Letting $y_1 = 1$ and $y_3 = 0$, gives another T-invariant

as **y1**=(1 1 0 0)$^{\tau}$

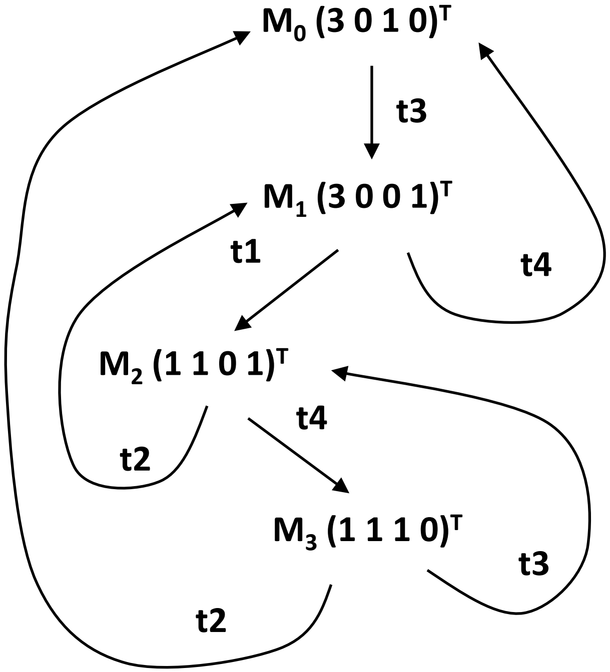
Also setting $y_1 = 0$ and $y_3 = 1$ gives a T-invariant

as **y2**=(0 0 1 1)$^{\tau}$

y1 and **y2** are two <u>independent</u> T-invariants. So are **y1** and **y;** and **y** and **y2;** but not **y, y1** and **y2** together.

Example: T-invariant

y=(1 1 1 1)$^{\tau}$; **y1**=(1 1 0 0)$^{\tau}$; **y2**=(0 0 1 1)$^{\tau}$



- T-invariant is filled with non-negative entries. It gives transition firing count to reach back to a previous marking. It shows the cyclic behavior of the PN.
- Limitations: It does not specify the order of transition firings
- For example for T-invariant: (1 1 1 1)$^{\tau}$, t1, t2, t3 and t4 fire in certain order to reach back to M0. (Order could be e.g., t3 t1 t2 t4)
- For example, T-invariants: (1100)$^{\tau}$ and (0011)$^{\tau}$ show cyclic behavior involving (t1, t2) and (t3, t4).

P-invariant

- A non-negative integer solution x of $A^T x=0$ is **P-invariant.**
- The change in marking due to transitions' firing can be represented in an equation.

$$M_{k+1} = M_k + A\mu$$

- By multiplying a transposed P-invariant x^T to both sides, we obtain:

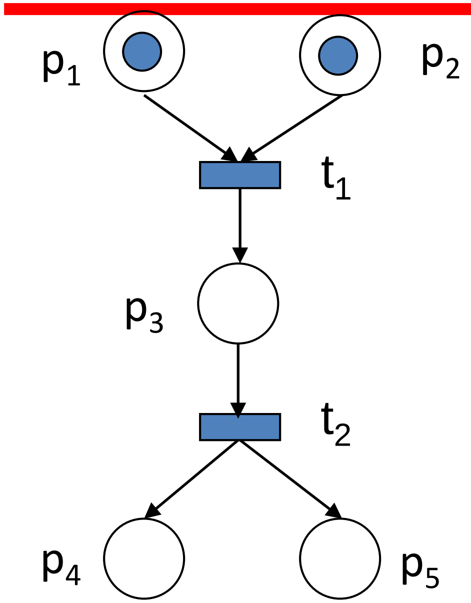
$$x^T M_{k+1} = x^T M_k + x^T A\mu$$

- If $A^T x=0$, i.e. $x^T A = 0$, we have

$$x^T M_{k+1} = x^T M_k$$

That means **weighted marking is unchanged**.

P-Invariant



$$\begin{array}{ll} (1\ 1\ 0\ 0\ 0) & m0 \\ \ \ \downarrow t1 & \\ (0\ 0\ 1\ 0\ 0) & m1 \\ \ \ \downarrow t2 & \\ (0\ 0\ 0\ 1\ 1) & m2 \end{array}$$

If $\mathbf{x}=[1\ 1\ 2\ 1\ 1]^T$

$x^T m_0 = x^T m_1 = x^T m_2 = 2$

$$A^T x = 0 \Rightarrow \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$$-x_1 - x_2 + x_3 = 0 \rightarrow x_3 = x_1 + x_2$$

$$-x_3 + x_4 + x_5 = 0 \rightarrow x_3 = x_4 + x_5$$

Independent P-invariants:

$[1\ 0\ 1\ 0\ 1]^T$

$[0\ 1\ 1\ 0\ 1]^T$

$[1\ 0\ 1\ 1\ 0]^T$

$[0\ 1\ 1\ 1\ 0]^T$

Positive p-invariants:

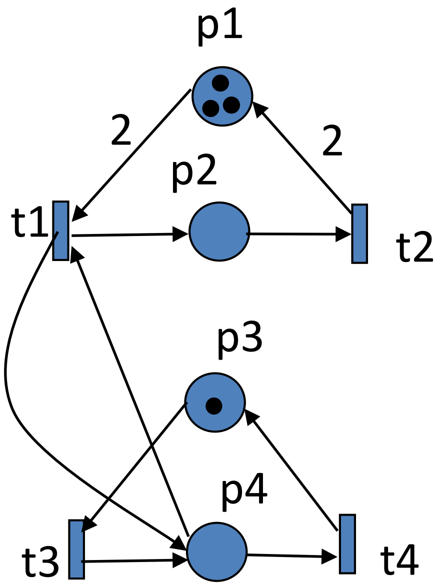
$[1\ \ 1\ \ 2\ \ 1\ \ 1]^T$

P-invariant

- P-invariants: set of places that retain the same number of tokens no matter what transition fires (conservation of post translational modification).
- The P-invariant can be explained intuitively in the following way:
  - The non-zero entries in a P-invariant represent weights associated with the corresponding places so that the weighted sum of tokens on these places is constant for all marking reachable from an initial marking.
  - The existence of a **positive** P-invariant (all the entries are positive) = **Conservativeness** where w=x. It further implies **boundedness** of a Petri net.

Example: calculating P-invariant

$$A^T x = 0 \quad \Rightarrow \quad \begin{pmatrix} -2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$



Solving the above gives

$-2x_1 + x_2 = 0 \rightarrow x_2 = 2x_1$

$x_4 = x_3$

Putting $x_1 = x_3 = 1$

We have a P-invariant is **x**=$(1\ 2\ 1\ 1\ )^T$ - a positive one

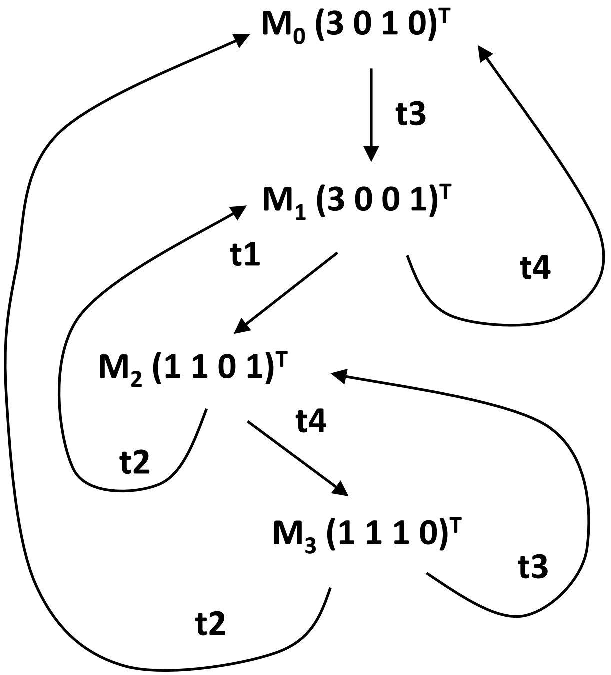
Letting $x_1 = 1$ and $x_3 = 0$ gives another P-invariant as **x1**=$(1\ 2\ 0\ 0)^T$

Also setting $x_1 = 0$ and $x_3 = 1$ gives a P-invariant as **x2**=$(0\ 0\ 1\ 1)^T$

x1 and **x2** are independent. So are **x** and **x1**; and **x** and **x2**; not but all three together.

Example: P-invariant

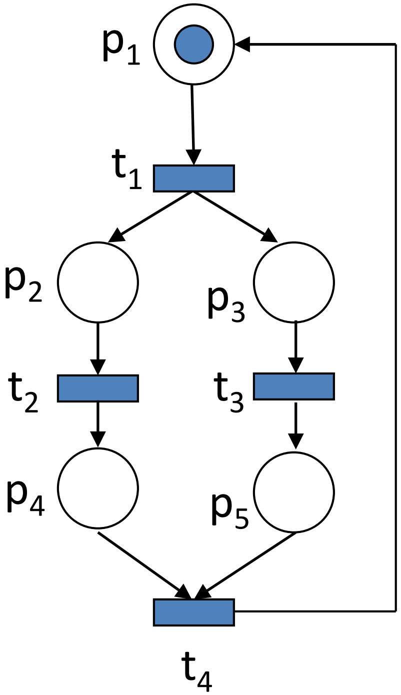
$x=(1\ 2\ 1\ 1\ )^{\tau}$; $x_1=(1\ 2\ 0\ 0\ )^{\tau}$; $x_2=(0\ 0\ 1\ 1)^{\tau}$

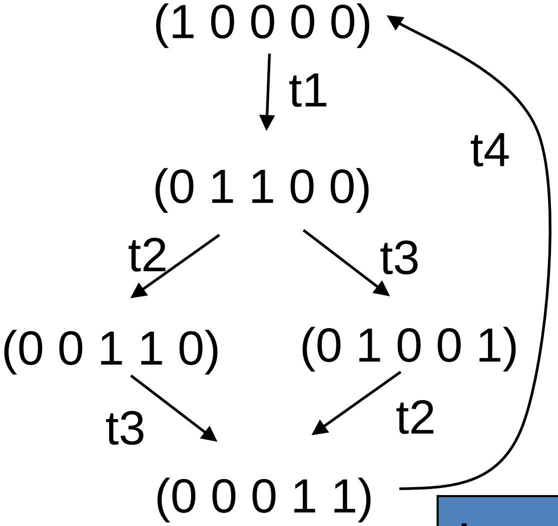


- If P-invariants contain **all positive numbers**, it shows that PN is **bounded**.
- Positive entries in a P-invariant gives weight on places and thus can be used to find the **conservativeness** property.
- e.g., using weights from P-invariants,
  - at M0: 1x3 + 2x0 + 1x1 + 1x0 = 4
  - at M1: 1x3 + 2x0 + 1x0 + 1x1 = 4
  - at M2: 1x1 + 2x1 + 1x0 + 1x1 = 4
  - at M3: 1x1 + 2x1 + 1x1 + 1x0 = 4

  which confirms conservativeness property.
- P-invariants having entries partly nonzero can be combined to give positive weights on places.
  - e.g., the second and third P-invariants can be combined to give the resultant equivalent to the first P-invariant.

Example: calculating P-invariant





$$A^T x = 0 \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$-x_1+x_2+x_3=0$ -> $x_1=x_2+x_3$

$-x_2+x_4=0$ -> $x_2=x_4$

$-x_3+x_5=0$ -> $x_3=x_5$

$x_1-x_4-x_5=0$ -> $x_1=x_4+x_5$

Let $x_2=x_4=1$ & $x_3=x_5=1$, we have $x_1=2$ => $\mathbf{x}=[2\ 1\ 1\ 1\ 1]^T$

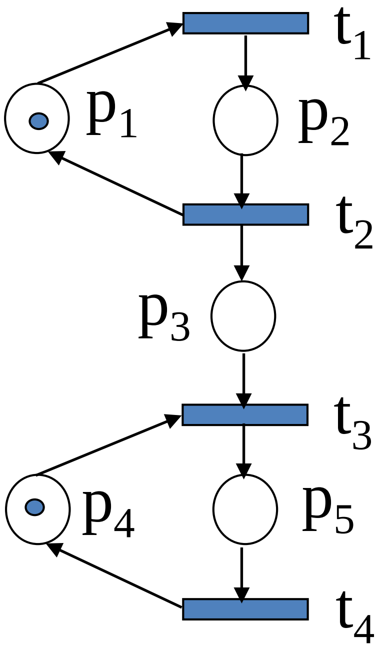
Let $x_2=x_4=1$ & $x_3=x_5=0$, we have $x_1=1$ => $\mathbf{x}_1=[1\ 1\ 0\ 1\ 0]^T$

Let $x_2=x_4=0$ & $x_3=x_5=1$, we have $x_1=2$ => $\mathbf{x}_2=[1\ 0\ 1\ 0\ 1]^T$

$\mathbf{x}_1$ & $\mathbf{x}_2$ are independent P-invariant.

$\mathbf{x}=[2\ 1\ 1\ 1\ 1]^T$ is positive P-invariant.

Example: Not Conservative



$$A^T x = 0 \rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$-x_1+x_2=0$ -> $x_1=x_2$

$x_1-x_2+x_3=0$ -> $x_2=x_1+x_3$

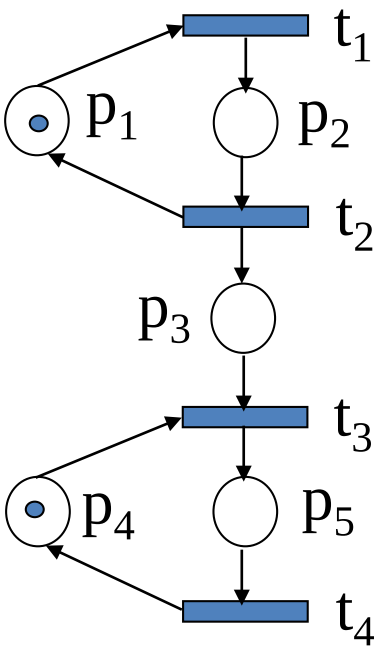
$-x_3-x_4+x_5=0$ -> $x_5=x_3+x_4$

$x_4-x_5=0$ -> $x_4=x_5$

$\mathbf{x}=[1\ 1\ 0\ 1\ 1]^T$

The entry for p3 is 0. so, the PN is not conservative; not bounded.

Example: Not Conservative



$m_0=(10010)^\tau$

$\downarrow t_1$

$m_1=(01010)^\tau$

$\downarrow t_2$

$m_2=(10110)^\tau$

Since $m_2 \geq m_0$ and $m_2(p_3) > m_0(p_3)$, ω replaces $m_2(p_3) = 1$.

$m_2=(10\ \omega 10)^\tau$

$t_1 \swarrow \qquad \searrow t_3$

Note: ω-1=ω

$m_3=(01\ \omega\ 10)^\tau$ $\qquad$ $m_4=(10\ \omega\ 01)^\tau$

From m_3: $t_2 \swarrow$ $(10\ \omega\ 10)^\tau = m_2$ $\qquad$ $t_3 \searrow$ $m_5=(01\ \omega\ 01)^\tau$

From m_4: $t_1 \swarrow$ $(01\ \omega\ 01)^\tau = m_5$ $\qquad$ $t_4 \searrow$ $(10\ \omega 10)^\tau = m_2$

From m_5: $t_2 \swarrow$ $(10\ \omega\ 01)^\tau = m_4$ $\qquad$ $t_4 \searrow$ $(01\ \omega\ 10)^\tau = m_3$

P and T Invariant Support

- The subset of places corresponding to the nonzero entries of a P-invariant is called its support, and denoted by ||x ||.
- The subset of transitions corresponding to the nonzero entries of a T-invariant is called its support, and denoted ||y||.

T Invariant Support

- T-Invariants:

  $y1=(1\ 1\ 0\ 0\ )^{\tau}$

  $y2=(1\ 1\ 1\ 1\ )^{\tau}$

  $y3=(0\ 0\ 1\ 1)^{\tau}$

- T-invariant support:

  $||y1|| = (t1, t2)$

  $||y2|| = (t1, t2, t3, t4\ )$

  $||y3|| = (t3,\ t4)$

P Invariant Support

- P-Invariants:

  x1=(1 1 1 1)

  x2=(1 1 0 0)

  x3=(0 0 1 1)

- P-invariant support:

  ||x1|| = (p1, p2, p3, p4)

  ||x2|| = (p1, p2)

  ||x3|| = (p3, p4)

Properties of P/T Invariants

- If α and β are two P or T-invariants (P/T-invariants), then given any non-zero integers k1 and k2, k1α + k2β is still a P/T-invariant.
- By using this property, you may derive a positive invariant given several P/T-invariants.